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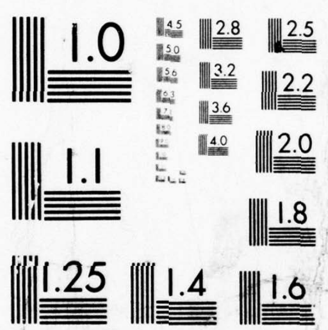

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Bounds for Solutions to a Class of Integrodifferential Equations Associated with a Theory of Rigid Nonconducting Material Dielectrics*

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Abstract

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Let H, H_+ be real Hilbert spaces with $H_+ \subseteq H$ algebraically and topologically and H_+ dense in H. Let H_- be the dual of H_+ via the inner product of H and denote by $L_S(H_+,H_-)$ the space of symmetric bounded linear operators from H_+ into H_-. We prove that the evolution of the electric displacement field in a simple class of holohedral isotropic dielectrics can be modeled by an abstract initial-value problem of the form

$$\underset{\sim}{u}_{tt} - \alpha \underset{\sim}{u}_t - \underset{\sim}{L}\underset{\sim}{u} + \int_0^t \underset{\sim}{M}(t-\tau)\underset{\sim}{u}(\tau)d\tau = \beta(t)\underset{\sim}{u}_o,\ 0 \le t \le T$$

$$\underset{\sim}{u}(0) = \underset{\sim}{u}_o,\ \underset{\sim}{u}_t(0) = \underset{\sim}{u}_1\ (\underset{\sim}{u}_o,\underset{\sim}{u}_1 \in H_+)$$

where $\underset{\sim}{L} \in L_S(H_+, H_-)$, $\underset{\sim}{M}(t) \in L^2([0,T]; L_S(H_+, H_-))$, $\beta(t) \in C^1([0,T])$, and α is an arbitrary (non-zero) real number. By employing a logarithmic convexity argument we derive growth estimates for solutions of the above system which lie in uniformly bounded classes of the form

$$N = \{\underset{\sim}{u} \in C^2([0,T]; H_+) \mid \sup_{[0,T)} ||\underset{\sim}{u}||_{H_+} \le N\}$$

for some N > 0; our results are derived under a variety of assumptions concerning α, $\beta(t)$, and the inital data (without making any definiteness assumptions

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on the operators L or $\underset{\sim}{M}(t)$, $0 \leq t < T$) and are used to obtain growth estimates for the electric displacement field $\underset{\sim}{D}(\underset{\sim}{x},t)$ in rigid dielectrics which satisfy constitutive relations of the form

$$\underset{\sim}{D}(\underset{\sim}{x},t) = a_o \underset{\sim}{E}(\underset{\sim}{x},t) + \int_0^t \phi(t-\tau)\underset{\sim}{E}(\underset{\sim}{x},\tau)d\tau$$

$$\underset{\sim}{H}(\underset{\sim}{x},t) = b_o \underset{\sim}{B}(\underset{\sim}{x},t) + \int_0^t \psi(t-\tau)\underset{\sim}{B}(\underset{\sim}{x},\tau)d\tau$$

where $\underset{\sim}{E}$, $\underset{\sim}{H}$, $\underset{\sim}{B}$ are the usual electromagnetic field variables, $(\underset{\sim}{x},t) \in \Omega \times [0,T)$, $\Omega \subseteq R^3$ is bounded region with smooth boundary $\partial\Omega$, a_o and b_o are positive constants, and ϕ, ψ are non-negative monotonically decreasing functions of t.

1. Introduction

In recent work [1] - [4] this author has derived stability and growth estimates for specific classes of solutions to initial-value problems associated with abstract integrodifferential equations of the form

$$\underset{\sim}{u}_{tt} - \underset{\sim}{N}\underset{\sim}{u} + \int_{-\infty}^{t} \underset{\sim}{K}(t-\tau)\underset{\sim}{u}(\tau)d\tau = \underset{\sim}{0}, \quad 0 \le t < T, \tag{1.1}$$

In this equation $\underset{\sim}{u} \in C^2([0,T); H_+)$ with $\underset{\sim}{u}_t \in C^1([0,T); H_+)$, and $\underset{\sim}{u}_{tt} \in C([0,T);H_-)$, where H_+, H_- are Hilbert spaces which are defined as follows: Let H be any real Hilbert space with inner-product $<,>$ and let $H_+ \subseteq H$ (algebraically and topologically) with H_+ dense in H; denote the inner-product on H_+ by $<,>_+$. Then H_- is the completion of H under the norm

$$||\underset{\sim}{w}||_- = \sup_{\underset{\sim}{v} \in H_+} \frac{|<\underset{\sim}{v},\underset{\sim}{w}>|}{||\underset{\sim}{v}||_+} \tag{1.2}$$

If we let $L(H_+,H_-)$ denote the space of bounded linear operators from H_+ into H_- then in (1.1) we only require that

(i) $\underset{\sim}{N} \in L(H_+,H_-)$ is symmetric and

(ii) $\underset{\sim}{K}(t)$, $\underset{\sim}{K}_t(t) \in L^2((-\infty,\infty); L(H_+,H_-))$

where $\underset{\sim}{K}_t$ denotes the strong operator derivative of $\underset{\sim}{K}$; no definiteness assumptions are placed on $\underset{\sim}{N}$ and thus the initial-value problem obtained by appending to (1.1) the initial data

$$\underset{\sim}{u}(0) = \underset{\sim}{f}, \ \underset{\sim}{u}_t(0) = \underset{\sim}{g}; \ \underset{\sim}{f},\underset{\sim}{g} \in H_+ \tag{1.3a}$$

and the prescription of the past history which is given by

$$\underset{\sim}{u}(\tau) = \underset{\sim}{U}(\tau), \ -\infty < \tau < 0 \tag{1.3b}$$

is, in general, non well-posed. If we restrict our attention to classes of bounded solutions to (1.1) - (1.3) of the form $N = \{\underset{\sim}{v} \in C^2([0,T); H_+) \mid \sup_{[0,T)} \|\underset{\sim}{v}(t)\|_+ \le N^2\}$ then it is possible to derive both stability and growth estimates for solutions $\underset{\sim}{u} \in N$ under the assumption that $\underset{\sim}{K}(0)$ satisfies

$$- \langle \underset{\sim}{v}, \underset{\sim}{K}(0)\underset{\sim}{v}\rangle \ge \kappa \|\underset{\sim}{v}\|_+^2, \quad \forall v \in H_+ \tag{1.4a}$$

where

$$\kappa \ge \omega T \sup_{[0,T)} \|\underset{\sim}{K}_t(t)\|_{L(H_+,H_-)} \tag{1.4b}$$

with ω the embedding constant for the injection $i: H \to H_+$.

The technique used in [1] - [3] is based on a logarithmic convexity argument first employed by Knops and Payne [5] for the abstract wave equation obtained from (1.1) by setting $\underset{\sim}{K}(t) \equiv 0$; a different logarithmic convexity argument was employed by this author in [4] to derive continuous data dependence theorems for the system (1.1), (1.3a), (1.3b). The results obtained in [2] - [4] are applied in those papers to obtain growth, stability, and continuous data dependence theorems for solutions to initial-value problems associated with the equations of motion for linear isothermal viscoelastic materials; the spaces H, H_+, and H_-, as well as the operators $\underset{\sim}{N}$ and $\underset{\sim}{K}(t)$, are constructed and no definiteness assumptions are made on the initial value of the relaxation tensor. In the case of a one-dimensional homogeneous (isothermal) linear viscoelastic body, it is shown in [3] that the conditions (1.4a), (1.4b) are equivalent to the requirement that

$$g'(0) \leq -\kappa \text{ with } \kappa > \omega T\left(\sup_{[0,T)} |\ddot{g}(t)|\right) \tag{1.5}$$

where $g(t)$ is the relaxation function of the material.

More recently we have turned our attention to the way in which integro-differential equations arise in the theory of polarized non-conducting material dielectrics, i.e., in [6] we have considered the following problem: Let $\underset{\sim}{E}$, $\underset{\sim}{B}$, $\underset{\sim}{P}$, and $\underset{\sim}{D}$ denote, respectively, the electric field vector, the magnetic flux density, the polarization vector, and the electric displacement in a non-conducting medium; the polarization and electric displacement vectors are related via

$$\underset{\sim}{D} = \varepsilon_o \underset{\sim}{E} + P, \ \varepsilon_o \equiv \text{const.} \tag{1.6}$$

If (x^i,t), $i = 1,2,3$, denotes a Lorentz reference frame, with the (x^i) rectangular Cartesian coordinates and t the time parameter, then Maxwell's equations have the local form

$$\frac{\partial \underset{\sim}{B}}{\partial t} + \text{curl } \underset{\sim}{E} = 0, \ \text{div } \underset{\sim}{B} = 0 \tag{1.7}$$

$$\text{curl } \underset{\sim}{H} - \frac{\partial \underset{\sim}{D}}{\partial t} = 0, \ \text{div } \underset{\sim}{D} = 0 \tag{1.8}$$

whenever the density of free current $\underset{\sim}{J}_F = \underset{\sim}{0}$, the magnetization $\underset{\sim}{M} = \underset{\sim}{0}$, and the density of free charge $Q_F = 0$; in (1.7b), $\underset{\sim}{H}$ represents the magnetic intensity and is related to the magnetic flux density via $\underset{\sim}{H} = \mu_o^{-1} \underset{\sim}{B}$ where $\varepsilon_o \mu_o = c^{-2}$, c being the speed of light in a vacuum. A determinate system of equations for the fields appearing in Maxwell's equations is obtained by specifying a set of constitutive relations. For example, in a vacuum $\underset{\sim}{P} = \underset{\sim}{0}$ so

$$\underset{\sim}{D} = \varepsilon_o \underset{\sim}{E}, \ \underset{\sim}{H} = \mu_o^{-1} \underset{\sim}{B} \tag{1.9}$$

while in a rigid, linear, stationary nonconducting dielectric

$$\underset{\sim}{D} = \underset{\sim}{\varepsilon} \cdot \underset{\sim}{E}, \ \underset{\sim}{B} = \underset{\sim}{\mu} \cdot \underset{\sim}{H} \tag{1.10}$$

where $\underset{\sim}{\varepsilon}$ and $\underset{\sim}{\mu}$ are constant second order tensors; the constitutive equations (1.10) were given by Maxwell in 1873 [7]. In [6] we considered the set of equations which define the dielectric as being a Maxwell-Hopkinson material, i.e., (1.10_2) and

$$\underset{\sim}{D}(t) = \varepsilon \underset{\sim}{E}(t) + \int_{-\infty}^{t} \phi(t-\tau) \underset{\sim}{E}(\tau) d\tau \tag{1.11}$$

where $\varepsilon > 0$ and $\phi(t)$ is a continuous monotonically decreasing function for $t \geq 0$; following a suggestion of Maxwell, Hopkinson [8] employed the constitutive equations (1.10_2), (1.11) in connnection with his studies on the residual charge of the Leyden jar. It was demonstrated in [6] that (1.11) in conjunction with the local Maxwell equations (1.7), (1.8), yield certain integrodifferential equations for the evolution of the electric field and the electric displacement field, respectively, in a non-conducting material dielectric of Maxwell-Hopkinson type.

By introducing suitable Hilbert spaces H, H_+, H_- and operators $N \in L(H_+, H_-)$ and $K(t) \in L^2((-\infty,\infty); L(H_+,H_-))$ we were able in [6] to treat the initial-boundary value problem for $\underset{\sim}{D}$, as a special case of the abstract initial-value problem (1.1), (1.2) (in [6] we assumed that $\underset{\sim}{D}(\tau) = \underset{\sim}{0}$, $-\infty < \tau < 0$). From the stability and growth estimates derived for the electric displacement field $\underset{\sim}{D}$, corresponding estimates were then derived for the electric field $\underset{\sim}{E}$ (1) by employing the relation

(1) For an excellent discussion of the qualitative behavior of electromagnetic fields and dielectric constants in dielectrics of Maxwell-Hopkinson type (especially in the presence of an applied time periodic electric field) we refer the reader to the monograph of H. Fröhlich, Theory of Dielectrics, Oxford U. Press (1949).

$$\underset{\sim}{E}(t) = \varepsilon^{-1}\underset{\sim}{D}(t) + \varepsilon^{-1} \int_0^t \Phi(t-\tau)\underset{\sim}{D}(\tau)d\tau \tag{1.12}$$

which is obtaining by inverting the Maxwell-Hopkinson relation (1.11) via the usual technique of successive approximation.

The constitutive relations associated with the Maxwell-Hopkinson theory, i.e., (1.10_2) and (1.11), embody three basic simplifying assumptions: they are linear, they effect an a priori separation of electric and magnetic effects, and they do not allow for magnetic memory effects. As early as 1912 Volterra [9] proposed extending the Maxwell-Hopkinson theory to treat the case where the dielectric is anisotropic, non-linear, and magnetized; his constitutive relations were of the form

$$\underset{\sim}{D}(\underset{\sim}{x},t) = \underset{\sim}{\varepsilon}\cdot\underset{\sim}{E}(\underset{\sim}{x},t) + \underset{\sim\infty}{\overset{t}{\mathcal{D}}} \left(\underset{\sim}{E}(\underset{\sim}{x},\tau)\right)) \tag{1.13a}$$

$$\underset{\sim}{B}(\underset{\sim}{x},t) = \underset{\sim}{\mu}\cdot\underset{\sim}{H}(\underset{\sim}{x},t) + \underset{\sim\infty}{\overset{t}{\mathcal{B}}} \left(\underset{\sim}{H}(\underset{\sim}{x},\tau)\right. \tag{1.13b}$$

and it can be shown that (1.13a) reduces to (1.11) if the functional $\underset{\sim}{\mathcal{D}}$ is linear and isotropic and the body satisfies various restrictions which follow from considerations of material symmetry. Of course, (1.13a), (1.13b) still effect an a priori separation of electric and magnetic effects and, as pointed out by Toupin and Rivlin [10], such a separation is inadequate with respect to predicting such a phenomena as the Faraday effect in dielectrics. In [10] Toupin and Rivlin postulated constitutive equations of the form

$$\underset{\sim}{D}(t) = \sum_{\nu=0}^{n} \underset{\sim}{a}_\nu \cdot \underset{\sim}{E}^{(\nu)}(t) + \sum_{\nu=0}^{n} \underset{\sim}{c}_\nu \cdot \underset{\sim}{B}^{(\nu)}(t) \tag{1.14a}$$

$$+ \int_{-\infty}^{t} \underset{\sim}{\phi}_1(t,\tau)\cdot\underset{\sim}{E}(\tau)d\tau + \int_{-\infty}^{t} \phi_2(t,\tau) \cdot \underset{\sim}{B}(\tau)d\tau$$

$$\underset{\sim}{H}(t) = \sum_{\nu=0}^{n} \underset{\sim\nu}{d}\cdot\underset{\sim}{E}^{(\nu)}(t) + \sum_{\nu=0}^{n} \underset{\sim\nu}{b} \cdot \underset{\sim}{B}^{(\nu)}(t) \tag{1.14b}$$

$$+ \int_{-\infty}^{t} \underset{\sim}{\psi}_1(t,\tau) \cdot \underset{\sim}{B}(\tau)d\tau + \int_{-\infty}^{t} \underset{\sim}{\psi}_2(t,\tau) \cdot \underset{\sim}{E}(\tau)d\tau$$

where $\underset{\sim}{E}^{(\nu)}(t) = d^{\nu}\underset{\sim}{E}(t)/dt^{\nu}$ and $\underset{\sim\nu}{a},\ldots, \underset{\sim\nu}{d}$ are constant tensors; the kernels $\underset{\sim}{\phi}_1,\ldots, \underset{\sim}{\psi}_2$ are taken to be continuous tensor functions of t and τ which satisfy growth conditions of the form

$$\phi_1(t,\tau) < \underset{\sim}{c}/(t-\tau)^{1+\rho}, \ \rho > 0$$

Toupin and Rivlin [10] also assumed that the dielectric does not exhibit aging and as a consequence it follows that $\underset{\sim}{D}(t)$ and $\underset{\sim}{H}(t)$ are periodic functions whenever $\underset{\sim}{E}(t)$ and $\underset{\sim}{B}(t)$ are; this latter result, when combined with the hypothesized growth estimates on the kernel functions, and early results of Volterra on the theory of functionals [9], yields the conclusion that $\underset{\sim}{\phi}_1,\ldots, \underset{\sim}{\psi}_2$ depend on t and τ only through the difference $t-\tau$ (the converse of this result is also true). Toupin and Rivlin [10] then prove that if the dielectric exhibits holohedral isotropy, i.e., if it admits as its group of material symmetry transformations the full orthogonal group, then $\underset{\sim}{E}(t)$ may be eliminated from (1.14b) and $\underset{\sim}{B}(t)$ may be eliminated from (1.14a); for a holohedral isotropic dielectric the constitutive equations (1.14a), (1.14b) reduce to

$$\underset{\sim}{D}(t) = \sum_{\nu=0}^{n} a_{\nu}\underset{\sim}{E}^{(\nu)}(t) + \int_{-\infty}^{t}\phi(t-\tau)\underset{\sim}{E}(\tau)d\tau \tag{1.15a}$$

$$\underset{\sim}{H}(t) = \sum_{\nu=0}^{n} b_{\nu}\underset{\sim}{B}^{(\nu)}(t) + \int_{-\infty}^{t}\psi(t-\tau)\underset{\sim}{B}(\tau)d\tau \tag{1.15b}$$

where $\phi = \phi_1$, $\psi = \psi_1$ and where (due to the assumption of holohederal isotropy) $\underset{\sim\nu}{a}$, $\underset{\sim\nu}{b}$, $\underset{\sim}{\phi}_1$ and $\underset{\sim}{\psi}_1$ are all proportional to the identity tensor and thus appear as scalars in (1.15a), (1.15b).

In this paper we examine the special case of (1.15a), (1.15b) which corresponds to the assumptions $a_\nu = 0$, $b_\nu = 0$, $\nu \ge 1$ and $\underset{\sim}{E}(\tau) = \underset{\sim}{0}$, $\underset{\sim}{B}(\tau) = \underset{\sim}{0}$, $-\infty < \tau < 0$, i.e.

$$\underset{\sim}{D}(t) = a_o \underset{\sim}{E}(t) + \int_0^t \phi(t-\tau)\underset{\sim}{E}(\tau)d\tau \tag{1.16a}$$

$$\underset{\sim}{H}(t) = b_o \underset{\sim}{B}(t) + \int_0^t \psi(t-\tau)\underset{\sim}{B}(\tau)d\tau \tag{1.16b}$$

This special case of a holohedral isotropic non-conducting material dielectric still embodies a separation of electric and magnetic effects in the constitutive theory but generalizes the Maxwell-Hopkinson theory in that magnetic memory effects are taken into account through the presence of the kernel function $\psi(t)$. In the next section we will formulate an initial-boundary value problem for the electric displacement field $\underset{\sim}{D}(t)$ in a holohedral isotropic dielectric; provided $\psi(0) \neq 0$, $\underset{\sim}{D}(t)$ will be shown to satisfy a (non-homogeneous) integrodifferential equation. By introducing suitable Hilbert spaces and operators, the initial-boundary value problem for $\underset{\sim}{D}(t)$ is easily demonstrated to be equivalent to an initial value problem for an abstract integrodifferential equation and growth estimates for specific classes of solutions to this abstract problem are then obtained by employing a suitable logarithmic convexity argument.

2. Initial-Boundary Value Problems for Holohedral Isotropic Dielectrics

Let (x^i,t) be a fixed Lorentz reference frame; the local forms of Maxwell's equations are then given by (1.7), (1.8). Let $\Omega \subseteq R^3$ be a bounded region with boundary $\partial\Omega$ and assume that $\partial\Omega$ is sufficiently smooth so that the divergence theorem may be applied. Finally, assume that Ω is filled with a holohedral isotropic non-conducting dielectric material which is non-deformable and which satisfies the hypotheses of §1 so that, in Ω, the electromagnetic field satisfies consitutive relations of the form (1.16a), (1.16b) where we assume that $a_o > 0$, $b_o > 0$ and $\phi(t)$, $\psi(t)$ are monotonically decreasing functions which are (at least) twice continuously differentiable on $[0,\infty)$ with $\psi^{(3)}(t)$ a bounded integrable function on $[0,\infty)$. The basic result of this section is

<u>Theorem II.1</u> The evolution of the electric displacement field $\underset{\sim}{D}(x,t)$ in any holohedral isotropic non-conducting material dielectric (which conforms to the constitutive hypotheses (1.16a), (1.16b)) is governed by the system of equations

$$\frac{\partial^2 D_i}{\partial t^2} + \Psi(0)\frac{\partial D_i}{\partial t} - b_o\dot{\Psi}(0)[c_o\delta_{ij}\delta_{j\ell}\frac{\partial^2 D_k}{\partial x_j\partial x_\ell} - D_i] \tag{2.1}$$

$$+ b_o\int_0^t(\ddot{\Psi}(t-\tau)D_i(\tau) - \Phi_o(t-\tau)\delta_{ik}\delta_{j\ell}\frac{\partial^2 D_k(\tau)}{\partial x_j\partial x_\ell})d\tau = b_o\dot{\Psi}(t)D_i(0)$$

where $c_o = 1/a_o\dot{\Psi}(0)$, $\Phi_o(t) = \Phi(t)/a_o$ and

$$\begin{aligned}\Phi(t) &= \sum_{n=1}^{\infty}(-1)^n\phi^n(t)\\ \phi^n(t) &= \int_0^t\phi^1(t-\tau)\phi^{n-1}(\tau)d\tau,\ n \geq 2\\ \phi^1(t) &= a_o^{-1}\phi(t)\end{aligned} \tag{2.2}$$

with an analogous defintion for $\Psi(t)$ in terms of $\psi(t)$.

Proof By using successive approximations we may invert the constitutive relations (1.16a) and (1.16b) to obtain, respectively,

$$\underset{\sim}{E}(t) = \frac{1}{a_o} \underset{\sim}{D}(t) + \frac{1}{a_o} \int_0^t \Phi(t-\tau)\underset{\sim}{D}(\tau)d\tau \tag{2.3a}$$

$$\underset{\sim}{B}(t) = \frac{1}{b_o} \underset{\sim}{H}(t) + \frac{1}{b_o} \int_0^t \Psi(t-\tau)\underset{\sim}{H}(\tau)d\tau \tag{2.3b}$$

with $\Phi(t)$ and $\Psi(t)$ defined in terms of $\phi(t)$ and $\psi(t)$, respectively, as indicated in (2.2). From (2.3a) and the second Maxwell relation in (1.8) div $\underset{\sim}{E}(t) = 0$ so

$$\underset{\sim\sim}{\Delta}\underset{\sim}{E}(t) = -\text{ curl curl } \underset{\sim}{E}(t) \tag{2.4}$$

From (2.3b), however, and the first Maxwell relation in (1.7)

$$\text{curl } \underset{\sim}{E}(t) = -\underset{\sim}{B}_t = -\frac{1}{b_o}\underset{\sim}{H}_t - \frac{1}{b_o}\Psi(0)\underset{\sim}{H}(t) - \int_0^t \Psi_t(t-\tau)\underset{\sim}{H}(\tau)d\tau \tag{2.5}$$

Therefore,

$$\begin{aligned} -\text{ curl curl } \underset{\sim}{E}(t) &= \frac{1}{b_o}(\text{curl } H)_t + \frac{1}{b_o}\Psi(0)(\text{curl } \underset{\sim}{H}(t)) \\ &+ \int_0^t \Psi_t(t-\tau)\text{curl } \underset{\sim}{H}(\tau)d\tau = \frac{1}{b_o}\underset{\sim}{D}_{tt} + \frac{1}{b_o}\Psi(0)\underset{\sim}{D}_t \\ &+ \int_0^t \Psi_t(t-\tau)\underset{\sim}{D}_\tau(\tau)d\tau \end{aligned} \tag{2.6}$$

where the second relation in (2.6) follows from the first Maxwell equation in (1.7). Combining (2.6_2) with (2.4) and employing (2.3a) we obtain

$$\mathbf{D}_{tt} + \Psi(0)\mathbf{D}_t + b_o \int_0^t \Psi_t(t-\tau)\mathbf{D}_\tau(\tau)d\tau = \frac{b_o}{a_o}\Delta\mathbf{D}(t) + \frac{b_o}{a_o}\int_0^t \Phi(t-\tau)\Delta\mathbf{D}(\tau)d\tau \tag{2.7}$$

However,

$$\int_0^t \Psi_t(t-\tau)\mathbf{D}_\tau(\tau)d\tau = \dot{\Psi}(0)\mathbf{D}(t) - \dot{\Psi}(t)\mathbf{D}(0) + \int_0^t \Psi_{\tau\tau}(t-\tau)\mathbf{D}(\tau)d\tau \tag{2.8}$$

Substituting (2.8) into (2.7) we have on $\Omega \times [0,\infty)$:

$$\begin{aligned} &\mathbf{D}_{tt} + \Psi(0)\mathbf{D}_t + b_o\dot{\Psi}(0)(\mathbf{I} - c_o\,\Delta)\mathbf{D}(t) \\ &\quad + b_o \int_0^t (\Psi_{tt}(t-\tau)\mathbf{I} - \Phi_o(t-\tau)\Delta)\mathbf{D}(\tau)d\tau = b_o\,\dot{\Psi}(t)\mathbf{D}(0). \end{aligned} \tag{2.9}$$

where $c_o = 1/a_o\ \dot{\Psi}(0)$ and $\Phi_o(t) = \Phi(t)/a_o$. Q. E. D.

In conjunction with he integrodifferential equation (2.9) we consider initial and boundary data of the form

$$\mathbf{D}(\mathbf{x}, 0) = \mathbf{D}_o(\mathbf{x}),\ \mathbf{D}_t(\mathbf{x},0) = \mathbf{D}_1(\mathbf{x}),\ \mathbf{x} \in \bar{\Omega} \tag{2.10a}$$

$$\mathbf{D}(\mathbf{x},t) = \mathbf{0},\ (\mathbf{x},t) \in \partial\Omega \times [0,\infty) \tag{2.10b}$$

where $\mathbf{D}_0$, $\mathbf{D}_1$ are continuous on $\bar{\Omega}$. At this point it is convenient to recast the initial-boundary value problem (2.9), (2.10a), (2.10b) as an initial-value problem for an integrodifferential equation in Hilbert space.(2) As in [6] we let $\mathbf{C}_o^\infty(\Omega)$ denote the set of three dimensional vector fields with compact support in Ω whose components are in $C_o^\infty(\Omega)$. We take $H = \mathbf{L}_2(\Omega)$, i.e. the completion of $\mathbf{C}_o^\infty(\Omega)$ under the norm induced by the inner product

$$\langle \mathbf{v},\mathbf{w}\rangle_{\mathbf{L}_2} \equiv \int_\Omega v_i w_i d\mathbf{x} \tag{2.11}$$

while the Hilbert space H_+ is taken to be $\mathbf{H}_o^1(\Omega)$ the completion of

(2) We specify, below, three spaces H, H_+, and H_- which are taken to be certain Sobolev spaces in the application and which satisfy certain mild requirements in the general development.

$C_o^\infty(\Omega)$ under the norm induced by the inner product

$$\langle \underset{\sim}{v}, \underset{\sim}{w} \rangle_{H_o^1} \equiv \int_\Omega \frac{\partial v_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} \, d\underset{\sim}{x} \tag{2.12}$$

Finally, $H_- = H^{-1}(\Omega)$, the Hilbert space obtained by completing $C_o^\infty(\Omega)$ under the norm

$$||\underset{\sim}{v}||_{H^{-1}} \equiv \sup_{\underset{\sim}{w} \in H_o^1} \left[\left| \int_\Omega v_i w_i \, d\underset{\sim}{x} \right| / \left(\int_\Omega \frac{\partial w_i}{\partial x_j} \frac{\partial w_i}{\partial x_j} \, d\underset{\sim}{x} \right)^{\frac{1}{2}} \right] \tag{2.13}$$

It is known that $H_o^1(\Omega) \subseteq L_2(\Omega)$ (both topologically and algebraically) and that H_o^1 is dense in L_2. We denote by ω the embedding constant for the inclusion map i: $H_o^1(\Omega) \to L_2(\Omega)$.

Operators $\underset{\sim}{L} \in \mathcal{L}(H_o^1, H^{-1})$ and $M(t) \in L^2((-\infty,\infty); \mathcal{L}(H_o^1, H^{-1}))$ are now defined as follows:

$$(\underset{\sim}{L}\underset{\sim}{v})_i \equiv b_o \dot{\Psi}(0)[c_o \delta_{ik} \delta_{j\ell} \frac{\partial^2 v_k}{\partial x_j \partial x_\ell} - \delta_{ij} v_j], \quad \underset{\sim}{v} \in H_o^1(\Omega) \tag{2.14a}$$

$$(\underset{\sim}{M}(t)\underset{\sim}{v})_i \equiv b_o[\ddot{\Psi}(t) \delta_{ij} v_j - \Phi_o(t) \delta_{ik} \delta_{j\ell} \frac{\partial^2 v_k}{\partial x_j \partial x_\ell}] \begin{cases} v \in H_o^1(\Omega) \\ t \in (-\infty,\infty) \end{cases} \tag{2.14b}$$

where the derivative are taken in the distribution sense. It follows directly from these definitions and the smoothness assumptions on $\phi(t)$ and $\chi(t)$ that

(i) $\underset{\sim}{L} \in \mathcal{L}_S(H_o^1, H^{-1})$, $\underset{\sim}{M}(t) \in \mathcal{L}_S(H_o^1, H^{-1})$, $t \in (-\infty,\infty)$

(ii) $\underset{\sim}{M}_t(\cdot) \in L^2((-\infty,\infty); \mathcal{L}(H_o^1, H^{-1}))$

where $L_S(H_o^1, H^{-1})$ denotes the space of all symmetric bounded linear operators from H_o^1 into H^{-1} and $\underset{\sim}{M}_t$ is the strong operator derivative of $\underset{\sim}{M}(\cdot)$. Thus the system (2.1), (2.10a), (2.10b) is equivalent to

$$\underset{\sim}{D}_{tt} + \Psi(0)\underset{\sim}{D}_t - \underset{\sim}{L}\underset{\sim}{D} + \int_0^t \underset{\sim}{M}(t-\tau)\underset{\sim}{D}(\tau)d\tau = b_o\dot{\Psi}(t)\underset{\sim}{D}_o \tag{2.15}$$

$$\underset{\sim}{D}(0) = \underset{\sim}{D}_o, \ \underset{\sim}{D}_t(0) = \underset{\sim}{D}_1 \tag{2.16}$$

where $\underset{\sim}{D}_0, \underset{\sim}{D}_1 \in H_o^1$ and $\underset{\sim}{D} \in C^2([0,\infty); H_o^1)$. Actually, we shall be interested in solutions of (2.15), (2.16) on finite time intervals of the form $[0,T)$ where T, $0 < T < \infty$, is an arbitrary real number; this suggests that we examine the following abstract initial-value problem: Let H, H_+ be Hilbert spaces with inner products $<,>$ and $<,>_+$, respectively, and assume that $H_+ \subseteq H$ (algebraically and topologically) with H_+ dense in H; define H_- as in (1.2). We consider solutions $\underset{\sim}{u} \in C^2([0,T); H_+)$ of the system

$$\underset{\sim}{u}_{tt} - \alpha\underset{\sim}{u}_t - \underset{\sim}{L}\underset{\sim}{u} + \int_0^t M(t-\tau)\underset{\sim}{u}(\tau)d\tau = \beta(t)\underset{\sim}{u}_o, \ 0 \le t < T \tag{2.17}$$

$$\underset{\sim}{u}(0) = \underset{\sim}{u}_o, \ \underset{\sim}{u}_t(0) = \underset{\sim}{u}_1 \ (\underset{\sim}{u}_o, \underset{\sim}{u}_1 \in H_+) \tag{2.18}$$

where $\alpha \neq 0$ is an arbitrary real constant, $\beta(t)$ is any real-valued function such that $\dot{\beta}(t)$ exists a.e. on $[0,T)$, $L \in L_S(H_+,H_-)$ and $\underset{\sim}{M}(\cdot)$, $\underset{\sim}{M}_t(\cdot) \in L^2([0,T); L_S(H_+,H_-))$. We assume that $\underset{\sim}{u}_t \in C^1([0,T); H_+)$ and $\underset{\sim}{u}_{tt} \in C([0,T); H_-)$.

In §3 we derive some growth estimates for solutions $\underset{\sim}{u}(t)$ of the system (2.17), (2.18), which lie in the set N. Our estimates will be obtained under various combinations of the following hypotheses:

$$\alpha \begin{Bmatrix} > 0 \\ < 0 \end{Bmatrix}, \quad \underset{\sim}{u}_o \begin{Bmatrix} = \underset{\sim}{0} \\ \neq \underset{\sim}{0} \end{Bmatrix} \text{ and } \beta(t) \begin{Bmatrix} = 0, \; 0 \le t < T \\ \not\equiv 0, \; \text{on } [0,T) \end{Bmatrix}$$

In §4 we apply our results to the system consisting of (2.1), (2.10a), and (2.10b); at no point in this work do we make any definiteness assumptions on the operators $\underset{\sim}{L}$ or $\underset{\sim}{M}(t)$, $t \in [0,T)$.

3. <u>Some Growth Estimates</u>

Let $K(t) = \frac{1}{2}||\underset{\sim}{u}_t||^2$ denote the kinetic energy associated with solutions $\underset{\sim}{u}$ of the system (2.17), (2.18) and $P(t) = -\frac{1}{2}\langle\underset{\sim}{u},\underset{\sim}{N}\underset{\sim}{u}\rangle$ the potential energy; then $E(t) \equiv K(t) + P(t)$ is the total energy. Let γ and t_o be arbitrary non-negative real numbers and define

$$F(t;\gamma,t_o) \equiv ||\underset{\sim}{u}(t)||^2 + \gamma(t+t_o)^2, \quad 0 \le t < T \tag{3.1}$$

The growth estimates in this section all follow from the following basic

<u>Lemma</u> Let $\underset{\sim}{u} \in N$ be any solution of (2.17), (2.18). Suppose that

$$-\langle\underset{\sim}{v}, \underset{\sim}{M}(0)\underset{\sim}{v}\rangle \le \kappa||\underset{\sim}{v}||_+^2, \quad \forall \underset{\sim}{v} \in H_+ \tag{3.2a}$$

with

$$\kappa \ge \gamma T \sup_{[0,T)} ||\underset{\sim}{M}_t||_{L(H_+,H_-)} \tag{3.2b}$$

Then there exists $\mu > 0$ such that for all t, $0 \le t < T$

$$FF'' - F'^2 \ge -2F(2E(0) + \mu) + \alpha FF' - 2\alpha F(\gamma(t+t_o) + 4\int_0^t K(\tau)d\tau) \tag{3.3}$$

$$+ 2F(2\int_0^t \dot{\beta}(\tau)\langle\underset{\sim}{u},\underset{\sim}{u}_o\rangle d\tau - \beta(t)\langle\underset{\sim}{u},\underset{\sim}{u}_o\rangle) + 4F\beta(0)||\underset{\sim}{u}_o||^2$$

<u>Proof</u> From the definition of $F(t;\gamma,t_o)$, i.e. (3.1), we compute

$$F'(t;\gamma,t_o) = 2\langle \underset{\sim}{u},\underset{\sim}{u}_t\rangle + \gamma(t+t_o) \tag{3.4}$$

$$F''(t;\gamma,t_o) = 2\|\underset{\sim}{u}_t\|^2 + 2\alpha\langle \underset{\sim}{u},\underset{\sim}{u}_t\rangle + 2\langle \underset{\sim}{u},\underset{\sim}{L}\underset{\sim}{u}\rangle \tag{3.5}$$
$$- 2\langle \underset{\sim}{u}, \int_0^t \underset{\sim}{M}(t-\tau)\underset{\sim}{u}(\tau)d\tau + 2\beta(t)\langle \underset{\sim}{u},\underset{\sim}{u}_o\rangle + 2\gamma,$$

where we have made use of (2.17) in (3.5). Using the definitions of $K(t)$, $E(t)$, we may rewrite (3.5) in the form

$$F''(t;\gamma,t_o) = 2\alpha\langle \underset{\sim}{u},\underset{\sim}{u}_t\rangle + 2\beta(t)\langle \underset{\sim}{u},\underset{\sim}{u}_o\rangle - 2\langle \underset{\sim}{u}, \int_0^t \underset{\sim}{M}(t-\tau)\underset{\sim}{u}(\tau)d\tau \tag{3.6}$$
$$+ 4(2K(t) + \gamma) - 2(2E(0) + \gamma) - 4(E(t) - E(0))$$

However, for any τ, $0 \le \tau \le t < T$

$$E'(\tau) = \langle \underset{\sim}{u}_\tau,\underset{\sim}{u}_{\tau\tau}\rangle - \langle \underset{\sim}{u}_\tau,\underset{\sim}{L}\underset{\sim}{u}\rangle = \alpha\|\underset{\sim}{u}_\tau\|^2 + \beta(\tau)\langle \underset{\sim}{u}_\tau,\underset{\sim}{u}_o\rangle \tag{3.7}$$
$$- \langle \underset{\sim}{u}_\tau, \int_0^\tau \underset{\sim}{M}(\tau-\sigma)u(\sigma)d\sigma\rangle$$

Therefore,

$$E'(\tau) = 2\alpha K(\tau) + \beta(\tau)\langle \underset{\sim}{u}_\tau,u_o\rangle - \frac{d}{d\tau}\langle \underset{\sim}{u}(\tau), \int_0^\tau M(\tau-\sigma)\underset{\sim}{u}(\sigma)d\sigma\rangle \tag{3.8}$$
$$+ \langle \underset{\sim}{u}(\tau), \int_0^\tau \underset{\sim}{M}_\tau(\tau-\sigma)\underset{\sim}{u}(\sigma)d\sigma\rangle + \langle \underset{\sim}{u}(\tau), \underset{\sim}{M}(0)\underset{\sim}{u}(\tau)\rangle$$

Integrating this last result from zero to t and substituting for $E(t) - E(0)$ in (3.6) we obtain

$$F''(t;\gamma,t_o) = 2\alpha\langle \underset{\sim}{u},\underset{\sim}{u}_t\rangle + 2\beta(t)\langle \underset{\sim}{u},\underset{\sim}{u}_o\rangle + 2\langle \underset{\sim}{u}, \int_0^t \underset{\sim}{M}(t-\tau)\underset{\sim}{u}(\tau)d\tau\rangle \tag{3.9}$$
$$+ 4(2K(t) + \gamma) - 2(2E(0) + \gamma) - 8\alpha\int_0^t K(\tau)d\tau - 4\int_0^t \beta(\tau)\langle \underset{\sim}{u}_\tau,\underset{\sim}{u}_o\rangle d\tau$$
$$- 4\int_0^t \langle \underset{\sim}{u}(\tau), \int_0^\tau \underset{\sim}{M}_\tau(\tau-\sigma)\underset{\sim}{u}(\sigma)d\sigma\rangle d\tau - 4\int_0^t \langle \underset{\sim}{u}(\tau), \underset{\sim}{M}(0)\underset{\sim}{u}(\tau)\rangle d\tau$$

Therefore,

$$FF'' - F'^2 = 4F(2K(t) + \gamma) - F'^2 + 2\alpha F(2E(0) + \gamma) + 2\alpha F(\langle \underset{\sim}{u}, \underset{\sim}{u}_t\rangle - 4\int_0^t K(\tau)d\tau) \tag{3.10}$$

$$+ 2F(\beta(t)\langle \underset{\sim}{u}, \underset{\sim}{u}_o\rangle - 2\int_0^t \beta(\tau)\langle \underset{\sim}{u}_\tau, \underset{\sim}{u}_o\rangle d\tau) + 2F\langle \underset{\sim}{u}, \int_0^t \underset{\sim}{M}(t-\tau)\underset{\sim}{u}(\tau)d\tau\rangle$$

$$+ 4F\int_0^t \langle \underset{\sim}{u}(\tau), \int_0^\tau \underset{\sim}{M}_\tau(\tau-\sigma)\underset{\sim}{u}(\sigma)d\sigma\rangle d\tau - 4F\int_0^t \langle \underset{\sim}{u}(\tau), \underset{\sim}{M}(0)\underset{\sim}{u}(\tau)\rangle d\tau$$

However, from (3.1), (3.4), the definition of $K(t)$, and the Schwarz inequality it follows that

$$G(t;\gamma,t_o) \equiv 4F(t;\gamma,t_o)(2K(t) + \gamma) - F'^2(t;\gamma,t_o) \geq 0 \tag{3.11}$$

and, therefore, (3.10) yields the inequality

$$FF'' - F'^2 \geq - 2F(2E(0) + \gamma) + \alpha F(\frac{d}{dt}||\underset{\sim}{u}||^2 - 8\int_0^t K(\tau)d\tau) \tag{3.12}$$

$$+ 2F(2\int_0^t \dot{\beta}(\tau)\langle \underset{\sim}{u}, \underset{\sim}{u}_o\rangle d\tau - \beta(t)\langle \underset{\sim}{u}, \underset{\sim}{u}_o\rangle) + 4F\beta(0)||\underset{\sim}{u}_o||^2 + 2F\langle \underset{\sim}{u}, \int_0^t \underset{\sim}{M}(t-\tau)\underset{\sim}{u}(\tau)d\tau\rangle$$

$$- 4F\int_0^t \langle \underset{\sim}{u}(\tau), \int_0^\tau \underset{\sim}{M}_\tau(\tau-\sigma)\underset{\sim}{u}(\sigma)d\sigma\rangle d\tau - 4F\int_0^t \langle \underset{\sim}{u}(\tau), \underset{\sim}{M}(0)\underset{\sim}{u}(\tau)\rangle d\tau$$

If we make note of the fact that

$$\frac{d}{dt}||\underset{\sim}{u}||^2 = F'(t;\gamma,t_o) - 2\gamma(t+t_o)$$

then we can rewrite (3.12) in the form

$$FF'' - F'^2 \geq - 2F(2E(0) + \gamma) + \alpha FF' - 2\alpha F(\gamma(t+t_o) + 4\int_0^t K(\tau)d\tau) \tag{3.13}$$

$$+ 2F(2\int_0^t \dot{\beta}(\tau)\langle \underset{\sim}{u}, \underset{\sim}{u}_o\rangle d\tau - \beta(t)\langle \underset{\sim}{u}, \underset{\sim}{u}_o\rangle) + 4F\beta(0)||\underset{\sim}{u}_o|| + 2F\langle \underset{\sim}{u}, \int_0^t \underset{\sim}{M}(t-\tau)\underset{\sim}{u}(\tau)d\tau\rangle$$

$$- 4F\int_0^t \langle \underset{\sim}{u}(\tau), \int_0^\tau \underset{\sim}{M}_\tau(\tau-\sigma)\underset{\sim}{u}(\sigma)d\sigma\rangle d\tau - 4F \int_0^t \langle \underset{\sim}{u}(\tau), \underset{\sim}{M}(0)\underset{\sim}{u}(\tau)\rangle d\tau$$

In order to complete the proof of the lemma we now use the hypotheses

(3.2a), (3.2b) and the fact that $u \in N$ to bound, from below, the sum of the last three terms in (3.13), i.e.

$$|<\underset{\sim}{u}, \int_0^t \underset{\sim}{M}(t-\tau)\underset{\sim}{u}(\tau)d\tau| \le ||\underset{\sim}{u}(t)||\int_0^t||\underset{\sim}{M}(t-\tau)\underset{\sim}{u}(\tau)||d\tau \tag{3.14a}$$

$$\le \omega||\underset{\sim}{u}(t)||_+ \int_0^t(||\underset{\sim}{M}(t-\tau)||_{L(H_+,H_-)})||\underset{\sim}{u}(\tau)||_+ d\tau$$

$$\le \omega T(\sup_{[0,T)}||\underset{\sim}{u}||_+)^2 \sup_{[0,T)}||\underset{\sim}{M}(t)||_{L(H_+,H_-)} \le \omega N^2 T \sup_{[0,T)}||\underset{\sim}{M}(t)||_{L(H_+,H_-)}$$

and thus, as $F(t;\gamma,t_o) \ge 0$, $0 \le t < T$,

$$2F<\underset{\sim}{u}, \int_0^t \underset{\sim}{M}(t-\tau)\underset{\sim}{u}(\tau)d\tau> \ge -2\omega N^2 T \sup_{[0,T)}||\underset{\sim}{M}(t)||_{L(H_+,H_-)}F(t;\gamma,t_o) \tag{3.14b}$$

Also,

$$-4F\int_0^t<\underset{\sim}{u}(\tau), \underset{\sim}{M}(0)\underset{\sim}{u}(\tau)>d\tau \ge 4\kappa F\int_0^t||\underset{\sim}{u}(\tau)||_+^2 d\tau \tag{3.15}$$

$$\ge 4\omega T \sup_{[0,T)}||\underset{\sim}{M}_t||_{L(H_+,H_-)}F\int_0^t||\underset{\sim}{u}(\tau)||_+^2 d\tau$$

by virtue of (3.2a) and (3.2b). Finally

$$\int_0^t <\underset{\sim}{u}(\tau), \int_0^\tau \underset{\sim}{M}_\tau(\tau-\sigma)\underset{\sim}{u}(\sigma)d\sigma>d\tau \le \int_0^t|<\underset{\sim}{u}(\tau), \int_0^\tau \underset{\sim}{M}_\tau(\tau-\sigma)\underset{\sim}{u}(\sigma)d\sigma>|d\tau \tag{3.16a}$$

$$\le \int_0^t||\underset{\sim}{u}(\tau)||(\int_0^\tau(||\underset{\sim}{M}_\tau(\tau-\sigma)||_{L(H_+,H_-)})||\underset{\sim}{u}(\sigma)||_+ d\sigma)d\tau$$

$$\le \omega \sup_{[0,T)}||\underset{\sim}{M}_t||_{L(H_+,H_-)}\int_0^t||\underset{\sim}{u}(\tau)||_+(\int_0^\tau||\underset{\sim}{u}(\sigma)||_+ d\sigma)d\tau$$

$$\le \omega \sup_{[0,T)}||\underset{\sim}{M}_t||_{L(H_+,H_-)}(\int_0^t||\underset{\sim}{u}(\tau)||_+ d\tau)^2$$

$$\le \omega T \sup_{[0,T)}||\underset{\sim}{M}_t||_{L(H_+,H_-)}\int_0^t||\underset{\sim}{u}(\tau)||_+^2 d\tau$$

from which we easily deduce that

$$- 4F\int_0^t \langle \underset{\sim}{u}(\tau), \int_0^\tau \underset{\sim}{M}_\tau(\tau-\sigma)\underset{\sim}{u}(\sigma)d\sigma\rangle d\tau \tag{3.16b}$$

$$\geq - 4\omega T \sup_{[0,T)} \|\underset{\sim}{M}_t\|_{L(H_+,H_-)} F\int_0^t \|\underset{\sim}{u}(\tau)\|_+^2 \, d\tau$$

Combining (3.13) with the estimates (3.14b), (3.15_2) *and* (3.16*b*) *we* obtain the estimate (3.3) with

$$\mu \equiv \gamma + \omega N^2 T \underset{[0,T)}{} \|\underset{\sim}{M}(t)\|_{L(H_+,H_-)} \qquad \text{(Q.E.D.)} \tag{3.17}$$

With the preceding Lemma *as a starting point we now begin our study* of the growth behavior of solutions to (2.17), (2.18) which lie in the class N; in each of the cases examined below we assume that $\underset{\sim}{M}(0)$ satisfies (3.2a) for some $\kappa > 0$ which satisfies (3.2b).

<u>Case I</u>: $\underset{\sim}{u}_o = 0$ and $\alpha < 0$

In this case $E(0) = \frac{1}{2}\|\underset{\sim}{u}_1\|^2$ and the second expression on the right-hand side of (3.3) is non-negative; thus

$$FF'' - F'^2 \geq - 2F(\|\underset{\sim}{u}_1\|^2 + \mu) - |\alpha|FF' \tag{3.18}$$

for all t, $0 \leq t < T$, where μ is given by (3.17). However, for γ, t_o arbitrary nonnegative real numbers,

$$\lambda\gamma t_0^2 \leq \lambda\|\underset{\sim}{u}(t)\|^2 + \lambda\gamma(t+t_o)^2 \equiv \lambda F(t;\gamma;t_o) \tag{3.19}$$

for any $\lambda \geq 0$. If, *in* particular, we choose

$$\lambda = \lambda(\gamma;t_o) \equiv 2(\|\underset{\sim}{u}_1\|^2 + \mu)/\gamma t_o^2 \tag{3.20}$$

then for all t, $0 \leq t < T$, and all γ, $t_o \geq 0$

$$2(||\underset{\sim}{u}_1||^2 + \mu) \leq \lambda(\gamma;t_o)F(t;\gamma,t_o) \tag{3.21}$$

and (3.18) may be replaced by the estimate

$$FF'' - F'^2 \geq -\lambda(\gamma;t_o)F^2 - |\alpha|FF' \tag{3.22}$$

The differential inequatlity (3.22) now forms the basis for the following growth estimate:

<u>Theorem III.1</u> Let $\underset{\sim}{u} \in N$ be any solution of (2.17), (2.18) with $\underset{\sim}{u}_o = \underset{\sim}{0}$ and $\alpha < 0$. Assume that $\underset{\sim}{M}(0)$ satisfies (3.2a), (3.2b) and that $T > 1/|\alpha|$. Then there exists a constant $M > 0$ such that

$$||\underset{\sim}{u}(t)||^2 \leq M^{1-\bar{\delta}} e^{-\frac{\bar{\lambda}}{|\alpha|} t}, \quad 0 \leq t < T \tag{3.23}$$

where $\bar{\delta}$ is given by (3.27).

<u>Proof</u> From (3.22) and Jensen's inequality we obtain the estimate,

$$F(t;\gamma,t_o) \leq e^{-\frac{\lambda}{|\alpha|} t}[F(t_1;\gamma,t_o)e^{\frac{\lambda}{|\alpha|} t_1}]^{\delta}[F(t_2;\gamma,t_o)e^{\frac{\lambda}{|\alpha|} t_2}]^{1-\delta} \tag{3.24}$$

(valid for $0 \leq t_1 < t \leq t_2 < T$) where

$$\delta(t) = (e^{-|\alpha|t} - e^{-|\alpha|t_2})/(e^{-|\alpha|t_1} - e^{-|\alpha|t_2}) \tag{3.25}$$

The interval $[t_1,t_2] \subseteq [0,T)$ is any closed interval such that $F(t;\gamma,t_o) > 0$, $t_1 \leq t \leq t_2$. However, it is a simple consequence of (3.24) and the defintion of $F(t;\gamma,t_o)$ that $F(t;\gamma,t_o) \equiv 0$ on $[0,T)$ if $F(\bar{t};\gamma,t_o) = 0$ for any $\bar{t} \in [0,T)$. Thus, without loss of generality, we may assume that $F(t;\gamma,t_o) > 0$, $0 \leq t < T$. Taking $t_1 = 0$, $t_2 = T$ in (3.14) we obtain

$$F(t;\gamma,t_o) \leq e^{\frac{-\lambda}{|\alpha|} t}[\gamma t_o^2]^{\bar{\delta}}[F(T;\gamma,t_o)e^{\frac{\lambda}{|\alpha|} T}]^{1-\bar{\delta}} \tag{3.26}$$

where

$$\bar{\delta}(t) = (e^{-|\alpha|t} - e^{-|\alpha|T})/(1 - e^{-|\alpha|T}) \tag{3.27}$$

We now choose $\gamma = 1/t_o^2$ and then take the limit in (3.26) as $t_o \to +\infty$. Clearly, as

$$F(t;\gamma 1/t_o^2, t_o) = ||\underset{\sim}{u}(t)||^2 + (\frac{t}{t_o} + 1)^2$$

$$\lim_{t_o \to +\infty} F(t;1/t_o^2,t_o) = ||\underset{\sim}{u}(t)||^2 + 1 \tag{3.28}$$

for all $t \in [0,T)$. Also, as $\underset{\sim}{u} \in \underset{\sim}{N}$

$$\lim_{t_o \to +\infty} F(T;1/t_o^2,t_o) = \lim_{t_o \to +\infty} (||\underset{\sim}{u}(T)||^2 + (\frac{T}{t_o} + 1)^2)) \le \omega^2 N^2 + 1 \tag{3.29}$$

$$\lim_{t_o \to +\infty} \lambda(1/t_o^2;t_o) = \lim_{t_o \to +\infty} 2(||\underset{\sim}{u}_1||^2 + 1/t_o^2 + \bar{\mu}) = 2(||\underset{\sim}{u}_1||^2 + \bar{\mu}) \equiv \bar{\lambda} \tag{3.30}$$

where $\bar{\mu} = \omega N^2 T \sup_{[0,T)} ||\underset{\sim}{M}(t)||_{L(H_+,H_-)}$. Thus, with $\gamma = 1/t_o^2$ and $t_o \to +\infty$ in (3.26), we obtain the estimate

$$||\underset{\sim}{u}(t)||^2 \le e^{-\frac{\bar{\lambda}}{|\alpha|} t} [(\omega^2 N^2 + 1)e^{\frac{\bar{\lambda}}{|\alpha|} T}]^{1-\bar{\delta}}, \quad 0 \le t < T, \tag{3.31}$$

and the result, which shows that $||\underset{\sim}{u}||^2$ is bounded above by an exponentially decreasing function of t for all $t \in [0,T)$, follows by choosing $M > 0$ so large that $\omega^2 N^2 + 1 < M \exp(-\bar{\lambda}/|\alpha|)$.

In contrast to the result contained in the statement of Theorem III.1, we have following theorem concerning lower bounds for solutions $\underset{\sim}{u} \in N$ of (2.17), (2.18).

Theorem III.2 Let $\underset{\sim}{u} \in N$ be any solution of (2.17), (2.18) with $\underset{\sim}{u}_o = 0$ and $\alpha < 0$ and assume that $\underset{\sim}{M}(0)$ satisfies (3.2a), (3.2b). If $|\alpha| < 1$ then there exists $T > 0$ such that $||\underset{\sim}{u}||^2$ is bounded below by a monotonically increasing exponential function of t, $0 \le t < T$.

Proof We bein by integrating the differential inequality (3.22) according to the "tangent property" of convex functions-assuming that $F(t;\gamma,t_o) > 0$, $0 \le t < T$, where $T > 0$ is an arbitrary real number; by the "tangent property" for convex functions we refer to the fact that the graph of a convex function[(2)] on $[0,T)$ lies above the tangent line to the curve at any point $\bar{t} \in [0,T)$. Thus, we obtain directly from (3.22) the estimate

$$F(t;\gamma,t_o) \ge F(0;\gamma;t_o)\exp\left[\left\{\frac{F'(0;\gamma,t_o) + \frac{\lambda}{|\alpha|}F(0;\gamma,t_o)}{|\alpha|F(0;\gamma,t_o)}\right\}(1-e^{-|\alpha|t}) - \frac{\lambda}{|\alpha|}t\right] \tag{3.32}$$

However, $F(0;\gamma,t_o) = \gamma t_o^2$ and $F'(0;\gamma,t_o) = 2\gamma t_o$. Therefore, if we set $\gamma = 1/t_o^2$ in (3.40) we obtain

$$||u(t)||^2 + [t/t_o + 1]^2 \ge \exp[\chi(t;t_o)],\ 0 \le t < T \tag{3.33}$$

where

$$\chi(t;t_o) \equiv \frac{1}{|\alpha|}\left[\left(\frac{2}{t_o} + \frac{\lambda(1/t_o^2;t_o)}{|\alpha|}\right)(1-e^{-|\alpha|t}) - \lambda(1/t_o^2;t_o)t\right] \tag{3.34}$$

and

$$\lambda(1/t_o^2;t_o) = 2\left(||\underset{\sim}{u}_1||^2 + \frac{1}{t_o^2} + \omega^2 N^2 T \sup_{[0,T)} ||\underset{\sim}{M}||_{L(H_+,H_-)}\right) \tag{3.35}$$

(2) The inequality (3.22) and the assumption that $F(t;\gamma,t_o) > 0$ on $[0,T)$ imply that $\ell n(F(\sigma;\gamma,t_o)e^{-\lambda/\alpha^2})$ is a convex function of $\sigma = e^{-|\alpha|t}$ on $[0,T)$.

We note, in passing, that $\chi(0;t_o) = 0$. For the sake of convenience we now set

$$\epsilon(t_o) = \frac{2}{t_o} + \frac{\lambda(1/t_o^2;t_o)}{|\alpha|}$$

Then

$$\chi'(t;t_o) = \epsilon(t_o)e^{-|\alpha|t} - \lambda(1/t_o^2;t_o) \tag{3.36}$$

From (3.36) it follows immediately that $\chi'(t;t_o) > 0$

for $0 < t < \frac{1}{|\alpha|} \ell n \left\{ \frac{\epsilon(t_o)}{\lambda(1/t_o^2;t_o)} \right\}$ provided $\epsilon(t_o) > \lambda(1/t_o^2;t_o)$. We now

take the limit in (3.33) as $t_o \to +\infty$ and obtain

$$||\underset{\sim}{u}(t)||^2 + 1 \geq \exp[\lim_{t_o \to +\infty} \chi(t;t_o)], \quad 0 \leq t < T \tag{3.37}$$

But

$$\lim_{t_o \to +\infty} \chi(t;t_o) = \frac{1}{|\alpha|}[\lim_{t_o \to +\infty} \epsilon(t_o)(1-e^{-|\alpha|t}) \tag{3.38}$$

$$- \lim_{t_o \to +\infty} \lambda(1/t_o^2;t_o)]$$

$$= \frac{\bar{\lambda}}{|\alpha|^2}(1-e^{-|\alpha|t}) - \bar{\lambda}t \equiv \bar{\chi}(t)$$

where $\bar{\lambda}$ is given by (3.30). Also

$$\lim_{t_o \to +\infty} \chi'(t;t_o) = \frac{d}{dt}\bar{\chi}(t) = \bar{\lambda}(\frac{e^{-|\alpha|t}}{|\alpha|} - 1) \tag{3.39}$$

and, therefore,

$$\bar{\chi}'(t) > 0, \quad 0 \leq t < \frac{1}{|\alpha|} \ell n(\frac{1}{|\alpha|}) \tag{3.40}$$

if $|\alpha| < 1$. The statement of the theorem now follows with $T = \frac{1}{|\alpha|} \ell n(\frac{1}{|\alpha|})$, i.e.,

$$||\underset{\sim}{u}(t)||^2 + 1 \geq \exp(\bar{\chi}(t)),\ 0 \leq t < \frac{1}{|\alpha|}\ \ell n(\frac{1}{|\alpha|}) \tag{3.41}$$

where $\bar{\chi}(t)$, as determined by (3.38), is nonnegative and monotonically increasing on $[0, \frac{1}{|\alpha|}\ \ell n(\frac{1}{|\alpha|}))$. Q.E.D.

<u>Case II</u>: $\underset{\sim}{u}_o = \underset{\sim}{0}$ and $\alpha > 0$

In this case the expression $H(t;\gamma;t_o) \equiv -\ 2\alpha F(\gamma(t+t_o) + 4\int_0^t K(\tau)d\tau$ can not be dropped from the differential inequality (3.3). As $t < T$ and $\alpha > 0$, (3.3) with $\underset{\sim}{u}_o = \underset{\sim}{0}$ implies that

$$FF'' - F'^2 \geq -\ 2F(||\underset{\sim}{u}_1||^2 + \mu) + \alpha FF' - 2\alpha F(\gamma(T+t_o) + 2\int_0^t ||\underset{\sim}{u}_\tau||^2 d\tau) \tag{3.42}$$

In order to proceed further we shall need the following

<u>Lemma</u> Let $\underset{\sim}{u} \in N$ be any solution of (2.17), (2.18) with $\underset{\sim}{u}_o = \underset{\sim}{0}$. Then there exists a real-valued continuous function $h_\alpha(t)$, defined for $0 \leq t < T$, such that

$$\frac{1}{2t}\int_0^t ||\underset{\sim}{u}_\tau||^2 d\tau \leq ||\underset{\sim}{u}_1||^2 + h_\alpha(T),\ 0 \leq t < T \tag{3.43}$$

<u>Proof</u>: From the identity

$$\underset{\sim}{u}_t = \int_0^t \underset{\sim}{u}_{\tau\tau}\ d\tau + \underset{\sim}{u}_1,$$

and (2.17), we obtain

$$\underset{\sim}{u}_t = \underset{\sim}{u}_1 + \alpha\underset{\sim}{u} + \int_0^t \underset{\sim}{L}\underset{\sim}{u}(\tau)d\tau - \int_0^t\int_0^\tau \underset{\sim}{M}(\tau-\sigma)\underset{\sim}{u}(\sigma)d\sigma d\tau \tag{3.44}$$

Thus,

$$||\underset{\sim}{u}_t|| \le ||\underset{\sim}{u}_1|| + \alpha||\underset{\sim}{u}(t)|| + \int_0^t ||\underset{\sim}{L}||_{L(H_+,H_-)} ||\underset{\sim}{u}(\tau)||_+ d\tau \tag{3.45}$$

$$+ \int_0^t \int_0^\tau ||\underset{\sim}{M}(t-\sigma)||_{L(H_+,H_-)} ||\underset{\sim}{u}(\sigma)||_+ d\sigma d\tau$$

$$\le ||\underset{\sim}{u}_1|| + \alpha\omega||\underset{\sim}{u}(t)||_+ + t||\underset{\sim}{L}||_{L(H_+,H_-)} \sup_{[0,T)} ||\underset{\sim}{u}(\tau)||_+$$

$$+ \frac{t^2}{2} \sup_{[0,t)} ||\underset{\sim}{M}(\tau)||_{L(H_+,H_-)} \sup_{[0,T)} ||\underset{\sim}{u}(\tau)||_+$$

$$\le ||\underset{\sim}{u}_1|| + p_\alpha(t) \sup_{[0,T)} ||\underset{\sim}{u}(\tau)||_+$$

where

$$p_\alpha(t) \equiv \alpha\omega + t||\underset{\sim}{L}||_{L(H_+,H_-)} + \frac{t^2}{2} \sup_{[0,T)} ||\underset{\sim}{M}(t)||_{L(H_+,H_-)} \tag{3.46}$$

Clearly $p_\alpha(t) < p_\alpha(T)$, for all $t \in [0,T)$ and, as $\underset{\sim}{u} \in N$

$$||\underset{\sim}{u}_1|| \le ||\underset{\sim}{u}_1|| + Np_\alpha(T), \quad 0 \le t < T \tag{3.47}$$

Therefore,

$$\int_0^t ||\underset{\sim}{u}_\tau||^2 d\tau \le 2t(||\underset{\sim}{u}_1||^2 + N^2 p_\alpha^2 (T)), \; 0 \le t < T \tag{3.48}$$

and the lemma follows with

$$h_\alpha(t) = N^2 p_\alpha^2(t) \tag{3.49}$$

If we combine (3.42) with (3.43) we obtain

$$FF'' - F'^2 \ge - 2F(||\underset{\sim}{u}_1||^2 + \tilde{\mu}) + \alpha FF' \tag{3.50}$$

where $\tilde{\mu} > 0$ is defined by

$$\tilde{\mu} = \mu + \alpha[\gamma(T+t_o) + 4T(||\underset{\sim}{u}_1||^2 + h_\alpha(T))] \tag{3.51}$$

Choosing

$$\lambda^* = \lambda^*(\gamma;t_o) = \frac{2(||\underset{\sim}{u}_1||^2 + \tilde{\mu})}{\gamma t_o^2} \tag{3.52}$$

we have

$$FF'' - F'^2 \geq -\lambda^*(\gamma;t_o)F^2 + \alpha FF', \; 0 \leq t < T \tag{3.53}$$

If we apply Jensen's inequality to (3.53) we obtain

$$F(t;\gamma,t_o) \leq e^{\frac{\lambda^* t}{\alpha}}[\gamma t_o^2]^{\delta^*}[F(T;\gamma,t_o)e^{\frac{\lambda^*}{\alpha}T}]^{1-\delta^*}, \; 0 \leq t < T \tag{3.54}$$

where

$$\delta^*(t) = (e^{\alpha t} - e^{\alpha T})/(1-e^{\alpha T}), \; 0 \leq t < T \tag{3.55}$$

Taking $\gamma = \frac{1}{t_o^2}$ in (3.54), extracting the limit as $t_o \to +\infty$, and then choosing $Q > 0$ so large that $\omega^2N^2 + 1 \leq Qe^{\frac{\bar{\lambda}^*}{\alpha}T}$ we obtain the estimate

$$||\underset{\sim}{u}(t)||^2 \leq Q^{1-*\delta}e^{\frac{\bar{\lambda}^*}{\alpha}t}, \; 0 \leq t < T \tag{3.56}$$

To close out our study of the case $\underset{\sim}{u}_o = 0$, $\alpha > 0$ we now integrate the differential inequality (3.53) according to the "tangent property" of convex functions and we obtain

$$F(t;\gamma,t_o) \geq \gamma t_o^2 \exp\left[\left\{\frac{2\gamma t_o - \frac{\lambda^*}{\alpha}\gamma t_o^2}{-\alpha\gamma t_o^2}\right\}(1-e^{\alpha t}) + \frac{\lambda^*}{\alpha}t\right] \tag{3.57}$$

which, with $\gamma = 1/t_o^2$, $\lambda^* = \lambda^*(1/t_o^2;t_o)$, reduces to

$$||\underset{\sim}{u}(t)||^2 + (\frac{t}{t_o} + 1)^2 \geq \exp\,[\{\frac{\lambda^*}{\alpha^2} - \frac{2}{\alpha t_o}\}\cdot(1-e^{\alpha t}) + \frac{\lambda^*}{\alpha}t] \tag{3.58}$$

Were we to follow the arguments previously employed we would, at this point, take the limit in (3.58) as $t_o \to +\infty$. This procedure, however, does not lead to a viable lower bound for $||\underset{\sim}{u}||^2$ in this case. It is worthwhile, however, to examine the function

$$J(t;\gamma,t_o) \equiv \left(\frac{\lambda^*(\gamma;t_o)}{\alpha^2} - \frac{2}{\alpha t_o}\right)\cdot(1-e^{\alpha t}) + \frac{\lambda^*(\gamma;t_o)t}{\alpha} \tag{3.59}$$

Clearly, $J(0;\gamma,t_o) = 0$ for arbitrary nonnegative constants γ, t_o. Also

$$J'(t;\gamma,t_o) = \left(\frac{2}{t_o} - \frac{\lambda^*(\gamma;t_o)}{\alpha}\right)e^{\alpha t} + \frac{\lambda^*(\gamma;t_o)}{\alpha} \tag{3.60}$$

from which, by the defintion of λ^*, it follows that

$$\left(\frac{\alpha\gamma t_o^2}{2}\right) J'(t;\gamma,t_o) = (k_1 + k_2\gamma)(1-e^{\alpha t}) + \alpha\gamma t_o \tag{3.61}$$

where

$$k_1 = ||\underset{\sim}{u}_1||^2(1 + 4\alpha T) + \bar{\mu} + 4\alpha T h_\alpha(T) \tag{3.62a}$$

$$k_2 = 1 + \alpha T \tag{3.62b}$$

Thus, if we choose

$$t_o = t_{o,\gamma} \equiv \frac{(k_1+k_2\gamma)}{\alpha\gamma}(e^{\alpha T} - 1), \quad \gamma > 0 \tag{3.63}$$

then $J'(t;\gamma,t_{o,\gamma}) > 0$ for all t, $0 \le t \le T$, and each real $\gamma > 0$, and we can state the following result:

<u>Theorem III.3</u> Let $\underset{\sim}{u} \in N$ be any solution of (2.17), (2.18) with $\underset{\sim}{u}_o = \underset{\sim}{0}$ and $\alpha > 0$ and assume that $\underset{\sim}{M}(0)$ satisfies (3.2a), (3.2b). Then for any $T > 0$ there exists $Q > 0$ such that $||\underset{\sim}{u}||^2$ satisfies (3.56) and, for each real $\gamma > 0$, $||\underset{\sim}{u}||^2$ also satisfies

$$||u(t)||^2 + \gamma(t+t_{o,\gamma})^2 \geq \gamma t_{o,\gamma}^2 \exp[J(t;\gamma,t_{o,\gamma})],\ 0 \leq t < T \tag{3.64}$$

where $t_{o,\gamma}$ is defined by (3.61a), (3.62b), and (3.63) and $J(t;\gamma,t_{o,\gamma})$, defined by (3.59) with $t_o = t_{o,\gamma}$, is nonnegative and strictly monotonically increasing on $[0,T)$.

The results obtained in cases I and II did not involve any hypotheses concerning the sign of the initial energy $E(0)$; as we assumed $\underset{\sim}{u}_o = \underset{\sim}{0}$ in both cases, $E(0) = \frac{1}{2}||\underset{\sim}{u}_1||^2 > 0$ if $\underset{\sim}{u}_1 \neq \underset{\sim}{0}$. In the cases considered below we remove the restriction that $\underset{\sim}{u}_o = \underset{\sim}{0}$.

<u>Case III</u>: $\underset{\sim}{u}_o \neq \underset{\sim}{0}$, $\alpha < 0$, and $\beta(t) = 0$, $0 \leq t < T$.

In this case (provided we use the fact that $\alpha < 0$ to delete the term $H(t;\gamma,t_o)$) inequality (3.3) reduces to

$$FF'' - F'^2 \geq -2F(||\underset{\sim}{u}_1||^2 - \langle \underset{\sim}{u}_0, \underset{\sim}{L}\underset{\sim}{u}_0\rangle + \mu) - |\alpha|FF' \tag{3.65}$$

with μ given by (3.17). We now assume that the initial data $\underset{\sim}{u}_0$, $\underset{\sim}{u}_1$ satisfies

$$||\underset{\sim}{u}_1||^2 - \langle \underset{\sim}{u}_0, \underset{\sim}{L}\underset{\sim}{u}_0\rangle < -\bar{\mu} \tag{3.66}$$

where $\bar{\mu} = \omega N^2 T \sup_{[0,T)} ||\underset{\sim}{M}(t)||_{L(H_+,H_-)}$. Taking $\gamma = 0$ in (3.65) we obtain ($F(t) = ||\underset{\sim}{u}(t)||^2$)

$$F(t)F''(t) - [F'(t)]^2 \geq -|\alpha|F(t)F'(t),\quad 0 \leq t < T, \tag{3.67}$$

Jensen's inequality then yields the upper bound

$$||\underset{\sim}{u}(t)||^2 \leq ||\underset{\sim}{u}_o||^{2\bar{\delta}}\ ||\underset{\sim}{u}(T)||^{2(1-\bar{\delta})},\ 0 \leq t < T \tag{3.68a}$$

We note that the hypothesis that $\underset{\sim}{u} \in N$, and (3.68), imply that there exists

$R > 0$ such that

$$||\underset{\sim}{u}(t)||^2 \leq R^{1-\bar{\delta}} ||\underset{\sim}{u}_o||^{2\bar{\delta}}, \quad 0 \leq t < T \tag{3.68b}$$

However, as (3.66) can not be valid for $||\underset{\sim}{u}_o||$ sufficiently small, (3.68b) represents only an upper bound on $||\underset{\sim}{u}(t)||$ in terms of $||\underset{\sim}{u}_o||$ and not a stability estimate. A better result is found by integrating (3.67) according to the "tangent property" of convex functions; in fact, directly from (3.32) with $\lambda = 0$ and $F(t;\gamma,t_o)$ replaced by $F(t) \equiv ||\underset{\sim}{u}(t)||^2$ we obtain

$$||\underset{\sim}{u}(t)||^2 \geq ||\underset{\sim}{u}_o||^2 \exp\left[\frac{2\langle\underset{\sim}{u}_1,\underset{\sim}{u}_o\rangle}{|\alpha|\,||\underset{\sim}{u}_o||^2}(1-e^{-|\alpha|t})\right], \quad 0 \leq t < T \tag{3.69}$$

From the estimate (3.69) it is obvious that if either $\langle\underset{\sim}{u}_o,\underset{\sim}{u}_1\rangle = 0$ or $\underset{\sim}{u}_1 = \underset{\sim}{0}$ (and $\langle\underset{\sim}{u}_o,\underset{\sim}{L}\underset{\sim}{u}_o\rangle > \bar{\mu}$) then $||\underset{\sim}{u}(t)||^2 \geq ||\underset{\sim}{u}_o||^2$ for all $t \in [0,T)$. On the other hand, if $\langle\underset{\sim}{u}_1,\underset{\sim}{u}_o\rangle > 0$, then on $[0,T)$, $||\underset{\sim}{u}(t)||^2$ is bounded below by a monotonically increasing exponential function of t. Finally if $\langle\underset{\sim}{u}_o,\underset{\sim}{u}_1\rangle < 0$ then $||\underset{\sim}{u}(t)||^2$ can not decay any faster than a monotonically decreasing exponential function of t. Our results are summarized as

Theorem III.4 Let $\underset{\sim}{u} \in N$ be any solution of (2.17), (2.18) with $\underset{\sim}{u}_o \neq 0$, $\alpha < 0$, and $\beta(t) \equiv 0$ on $[0,T)$. Assume that $M(0)$ satisfies (3.2a) and (3.2b). Then

(A) If the initial data satisfy (3.66), $||\underset{\sim}{u}(t)||$ is bounded above by $||\underset{\sim}{u}_o||$ according to (3.68b), $0 \leq t < T$

(B) If the initial data satisfy (3.66) then there exists $K(\alpha)$ such that for all t, $0 \leq t < T$,

$$||\underset{\sim}{u}(t)||^2 \geq ||\underset{\sim}{u}_o||^2 \exp[K(\alpha)(1-e^{-|\alpha|t})], \tag{3.70}$$

where for each real α, $K(\alpha)$ is real-valued and

(i) $K(\alpha) = 0$ if either $\underset{\sim}{u}_1 = \underset{\sim}{0}$ or $\langle \underset{\sim}{u}_o, \underset{\sim}{u}_1 \rangle = 0$

(ii) $K(\alpha) > 0$ if $\langle \underset{\sim}{u}_o, \underset{\sim}{u}_1 \rangle > 0$

(iii) $K(\alpha) < 0$ if $\langle \underset{\sim}{u}_o, \underset{\sim}{u}_1 \rangle < 0$

and

(iv) $|K(\alpha)| \to 0$ as $|\alpha| \to \infty$.

<u>Remark</u> The case $\underset{\sim}{u}_o \neq \underset{\sim}{0}$, $\alpha > 0$, and $\beta(t) \equiv 0$ can be treated in the same manner as Case III; in fact, from (3.50) (which was derived under the assumption that $\underset{\sim}{u}_o = \underset{\sim}{0}$ with $\alpha > 0$) we can write down immediately the differential inequality

$$FF'' - F'^2 \geq -2F(||\underset{\sim}{u}_1||^2 - \langle \underset{\sim}{u}_o, \underset{\sim}{L}\underset{\sim}{u}_o \rangle + \tilde{\mu}) + \alpha FF' \qquad (3.71)$$

for the case where $\underset{\sim}{u}_o \neq \underset{\sim}{0}$, $\alpha > 0$, but $\beta(t) \equiv 0$; in (3.71) $\tilde{\mu}$ is defined by (3.51). Suppose we set $\gamma = 0$; then if the initial data satisfy

$$(1 + 4\alpha T)||\underset{\sim}{u}_1||^2 - \langle \underset{\sim}{u}_o, \underset{\sim}{L}\underset{\sim}{u}_o \rangle \leq -(\bar{\mu} + 4\alpha T h_\alpha(T))$$

the above differential inequality reduces to

$$F(t)F''(t) - [F'(t)]^2 \geq \alpha F(t)F'(t), \quad 0 \leq t < T, \qquad (3.72)$$

where $F(t) = ||\underset{\sim}{u}(t)||^2$. We leave the integration of (3.72) and the analysis of the resulting estimates on $||\underset{\sim}{u}(t)||^2$ to the reader and turn, instead, to consider a case where both $\underset{\sim}{u}_o \neq 0$ and $\beta(t) \not\equiv 0$.

<u>Case IV</u> $\underset{\sim}{u}_o \neq \underset{\sim}{0}$, $\beta(t) \not\equiv 0$, $\alpha < 0$ and $\beta(0) > 0$

In this case (3.3) is easily seen to imply that

$$FF'' - F'^2 \geq -2F(2E(0) + \mu) - |\alpha| FF' \tag{3.73}$$

$$+ 2F(2\int_0^t \dot{\beta}(\tau)<\underset{\sim}{u},\underset{\sim}{u}_o>d\tau - \beta(t)<\underset{\sim}{u},\underset{\sim}{u}_o>) + 4F\beta(0)||\underset{\sim}{u}_o||^2$$

$$= -2F(2E(0) - 2\beta(0)||\underset{\sim}{u}_o||^2 + \mu) - |\alpha|FF'$$

$$+ 2F(2\int_0^t \dot{\beta}(\tau)<\underset{\sim}{u},\underset{\sim}{u}_o>d\tau - \beta(t)<\underset{\sim}{u},\underset{\sim}{u}_o>)$$

In order to proceed further we must bound from below the third expression on the right-hand side of the differential inequality (3.73_2); this is accomplished by the following lemma:

Lemma Suppose that $\dot{\beta}(t)$ is bounded on $[0,T)$ for each fixed T, $0 < T < \infty$. Then there exists a constant $C > 0$ such that

$$2\int_0^t \dot{\beta}(\tau)<\underset{\sim}{u},\underset{\sim}{u}_o>d\tau - \beta(t)<\underset{\sim}{u},\underset{\sim}{u}_o> \geq -C||\underset{\sim}{u}_o||, \quad 0 \leq t < T \tag{3.74}$$

Proof We set $\rho = \sup\limits_{[0,T)} |\dot{\beta}(t)| < \infty$. Then

$$\left|\int_0^t \dot{\beta}(\tau)<\underset{\sim}{u}(\tau), \underset{\sim}{u}_o>d\tau\right| = \left|<\int_0^t \dot{\beta}(\tau)\underset{\sim}{u}(\tau)d\tau, \underset{\sim}{u}_o>\right| \tag{3.75}$$

$$\leq (\int_0^t \left|\dot{\beta}(\tau)\right| ||\underset{\sim}{u}(\tau)||d\tau)||\underset{\sim}{u}_o||$$

$$\leq \rho(\int_0^T ||\underset{\sim}{u}(\tau)||d\tau)||\underset{\sim}{u}_o|| \leq \rho\omega NT||\underset{\sim}{u}_o||$$

so

$$\int_0^t \dot{\beta}(\tau)<\underset{\sim}{u},\underset{\sim}{u}_o>d\tau \geq -\rho\omega NT||\underset{\sim}{u}_o||, \quad 0 \leq t < T \tag{3.76}$$

Also

$$|\beta(t)<\underset{\sim}{u},\underset{\sim}{u}_o>| \leq |\beta(t)|\cdot|<\underset{\sim}{u},\underset{\sim}{u}_o>| \leq \omega N|\beta(t)|\cdot||\underset{\sim}{u}_o||$$

$$\leq \omega N \left| \int_0^t \dot{\beta}(\tau)d\tau + \beta(0)\right| ||\underset{\sim}{u}_o|| \leq \omega N(\rho T + \beta(0))||\underset{\sim}{u}_o||$$

so

$$-\beta(t)\langle \underset{\sim}{u},\underset{\sim}{u}_o\rangle \geq -\omega N(\rho T+\beta(0))||\underset{\sim}{u}_o||,\ 0\leq t<T \tag{3.78}$$

Combining (3.76) and (3.78) we obtain (3.74) with

$$C=\omega N(3\rho T+\beta(0))>0 \tag{3.79}$$

We now return to (3.73_2); in view of the last lemma this latter inequality implies that

$$FF''-F'^2\geq -2F(||\underset{\sim}{u}_1||^2+\textstyle\sum(\underset{\sim}{u}_o)+\mu)-|\alpha|FF' \tag{3.80}$$

where $\sum: H_+ \to R^+$ is defined by

$$\textstyle\sum(\underset{\sim}{w}) = 2\beta(0)||\underset{\sim}{w}||(\frac{C}{2\beta(0)}-||\underset{\sim}{w}||)-\langle \underset{\sim}{w},\underset{\sim}{L}\underset{\sim}{w}\rangle,\quad \underset{\sim}{w}\in H_+ \tag{3.81}$$

If we set $\gamma = 0$ then (3.80) reduces to

$$F(t)F''(t)-[F'(t)]^2\geq -2F(t)(||\underset{\sim}{u}_1||^2+\textstyle\sum(\underset{\sim}{u}_o)+\bar{\mu})-|\alpha|F(t) \tag{3.82}$$

with $F(t) = ||\underset{\sim}{u}(t)||^2$ and $\bar{\mu} = \omega N^2 \sup\limits_{[0,T)} ||\underset{\sim}{M}(t)||_{L(H_+,H_-)}$

and we have the following result:

<u>Theorem III.5</u> Let $\underset{\sim}{u}\in N$ be any solution of (2.17), (2.18) where $\underset{\sim}{u}_o \neq 0$, $\beta(t)\not\equiv 0$, $\alpha<0$, and $\beta(0)>0$. Assume that $\underset{\sim}{M}(0)$ satisfies (3.2a), (3.2b) and that $\dot{\beta}(t)$ is bounded for $0\leq t<T$. Then if the initial data satisfy

$$||\underset{\sim}{u}_1||^2+\textstyle\sum(\underset{\sim}{u}_o)\leq -\bar{\mu}, \tag{3.83}$$

where $\sum$ is defined by (3.81), $||\underset{\sim}{u}(t)||$ satisfies the estimates (3.68) and (3.69). In particular, if $\underset{\sim}{u}_1 = 0$ and $\sum(\underset{\sim}{u}_o)\leq -\bar{\mu}$ then

$||\underset{\sim}{u}(t)||^2 \geq ||\underset{\sim}{u}_o||^2$ for all t, $0 \leq t < T$.

Remark We leave for the reader the consideration of the other cases possible when $\underset{\sim}{u}_o \neq \underset{\sim}{0}$ and $\beta(t) \not\equiv 0$, e.g., $\alpha < 0$ and $\beta(0) \leq 0$; the stability and growth estimates which apply in these situations may easily be derived by suitably modifying the last lemma and making use of the basic differential inequalities derived for the previous cases.

4. Applications to Bounds for Electric Displacement Fields

In order to apply the results of the previous section to solutions of the initial-boundary value problem (2.1), (2.10a), (2.10b) (associated with the constitutive relations (1.16a), (1.16b)) we must delineate the form assumed by the basic hypothesis (3.2a), (3.2b). In other words, for the operator $\underset{\sim}{M}(t)$, which is defined by (2.14b), we wish to examine the implications of the requirement that

$$- \langle \underset{\sim}{v}, \underset{\sim}{M}(0)\underset{\sim}{v}\rangle_{L_2} \geq \kappa ||\underset{\sim}{v}||^2_{H^1_o}, \quad \underset{\sim}{v} \in H^1_o(\Omega) \tag{4.1}$$

with $\kappa \geq \omega T \sup_{[0,T)} ||\underset{\sim}{M}_t||_{L(H^1_o,\, H^{-1})}$. From (2.14b) and (2.11) we easily compute

$$\begin{aligned}
\langle \underset{\sim}{v}, \underset{\sim}{M}(0)\underset{\sim}{v}\rangle_H &= - \int_\Omega (\underset{\sim}{M}(0)\underset{\sim}{v})_i v_i d\underset{\sim}{x} = - b_o \ddot{\Psi}(0) \int_\Omega \delta_{ij} v_i v_j d\underset{\sim}{x} \\
&\quad + \frac{b_o}{a_o} \Phi(0) \int_\Omega \delta_{ik}\delta_{j\ell} \frac{\partial^2 v_k}{\partial x_j \partial x_\ell} v_i d\underset{\sim}{x} \\
&= - b_o \ddot{\Psi}(0) ||\underset{\sim}{v}||^2_H + \frac{b_o}{a_o} \Phi(0) \int_\Omega \delta_{ik}\delta_{j\ell} \frac{\partial^2 v_k}{\partial x_j \partial x_\ell} v_i d\underset{\sim}{x}
\end{aligned}$$

for any $\underset{\sim}{v} \in H^1_o$. But if $\underset{\sim}{v} \in H^1_o$ then

$$\int_\Omega \delta_{ik}\delta_{j\ell} \frac{\partial^2 v_k}{\partial x_j \partial x_\ell} v_i d\underset{\sim}{x} = \int_\Omega \delta_{j\ell} v_k \frac{\partial^2 v_k}{\partial x_j \partial x_\ell} d\underset{\sim}{x} \tag{4.3}$$

$$= - \int_\Omega \delta_{j\ell} \frac{\partial v_k}{\partial x_j} \frac{\partial v_k}{\partial x_\ell} d\underset{\sim}{x} = - ||\underset{\sim}{v}||^2_{H^1_o}$$

where we have used integration by parts together with the fact that $\underset{\sim}{v}$ vanishes on $\partial\Omega$[3]. Thus

$$- \langle \underset{\sim}{v}, \underset{\sim}{M}(0)\underset{\sim}{v} \rangle = - b_o \ddot{\Psi}(0) ||\underset{\sim}{v}||^2_H - \frac{b_o}{a_o} \Phi(0) ||\underset{\sim}{v}||^2_{H_+} \tag{4.4}$$

$$\geq - b_o (\omega^2 |\ddot{\Psi}(0)| + \frac{1}{a_o} \Phi(0)) ||\underset{\sim}{v}||^2_{H_+}$$

Therefore, (4.1a) will be satisfied if

$$- b_o (\omega^2 |\ddot{\Psi}(0)| + \frac{1}{a_o} \Phi(0)) \geq \kappa \tag{4.5}$$

with $\kappa \geq \omega T \sup_{[0,T)} ||\underset{\sim}{M}_t||_{L(H_+,H_-)}$. For the sake of convenience we now set $T(t) = \ddot{\Psi}(t)$. From (2.14b) again we have,

$$(\underset{\sim}{M}_t \underset{\sim}{v})_i = b_o [\dot{T}(t)\delta_{ij} v_j - \frac{\dot{\Phi}(t)}{a_o} \delta_{ik}\delta_{j\ell} \frac{\partial^2 v_k}{\partial x_j \partial x_\ell}] \text{ , } \underset{\sim}{v} \in H^1_o \tag{4.6}$$

so

$$|\langle \underset{\sim}{v}, \underset{\sim}{M}_t \underset{\sim}{v} \rangle_H| = |\int_\Omega [\underset{\sim}{M}_t v]_i v_i dx| \tag{4.7}$$

$$= |b_o \dot{T}(t) ||\underset{\sim}{v}||^2_H - \frac{b_o}{a_o} \dot{\Phi}(t) \int_\Omega \delta_{j\ell} v_k \frac{\partial^2 v_k}{\partial x_j \partial x_\ell} d\underset{\sim}{x}|$$

$$= b_o |\dot{T}(t)| ||\underset{\sim}{v}||^2_H + \frac{1}{a_o} \dot{\Phi}(t) ||\underset{\sim}{v}||^2_{H_+}$$

$$\leq b_o (\omega^2 |\dot{T}(t)| + \frac{1}{a_o} |\dot{\Phi}(t)|) ||\underset{\sim}{v}||^2_{H_+}$$

(3) This follows from the definition of H^1_o and a standard trace theorem.

It now follows that for each t, $0 \le t < T$,

$$||\underset{\sim}{M}_t||_{L(H_+,H_-)} = \sup_{\underset{\sim}{v}\in H_+} \frac{|\langle \underset{\sim}{v}, \underset{\sim}{M}_t \underset{\sim}{v}\rangle|}{||\underset{\sim}{v}||^2_{H_+}} \le b_o(\omega^2|\dot{T}(t)| + \frac{1}{a_o}|\dot{\Phi}(t)|) \tag{4.8}$$

Thus, (4.1b) will be satisfied if

$$\kappa \ge \omega T b_o(\omega^2 \sup_{[0,T)} |\dot{T}(t)| + \frac{1}{a_o} \sup_{[0,T)} |\dot{\Phi}(t)|) \tag{4.9}$$

Combining (4.5) and (4.9) we find that a condition which insures the validity of (4.1) is

$$-(\omega^2|T(0)| + \frac{1}{a_o}\Phi(0)) \ge \omega T(\omega^2 \sup_{[0,T)} |\dot{T}(t)| + \frac{1}{a_o} \sup_{[0,T)} |\dot{\Phi}(t)|) \tag{4.10}$$

It is clear, from (4.10), that this inequality can be satisfied only if $\Phi(0) < 0$ with $|\Phi(0)| > a_o\omega^2|T(0)|$. It is worthwhile, at this point, to recall the following result which has been proven in [6]:

<u>Lemma</u> Let $\phi(t) \in C^1[0,T)$ and assume that the series defining $\Phi(t)$ as well as the derived series, which is obtained by term by term differentiation, are uniformly convergent on every interval $[0,T-\varepsilon]$, $0 < \varepsilon < T$. If $\sup_{[0,T)} |\phi(t)| < a_o/T$ then

$$\text{(i)} \quad \sup_{[0,T)} |\Phi(t)| \le F(T) \tag{4.11}$$

$$\text{(ii)} \quad \sup_{[0,T)} |\dot{\Phi}(t)| \le \frac{F(T)}{T}\left\{1 + T\frac{\sup_{[0,T)} |\dot{\phi}(t)|}{\sup_{[0,T)} |\phi(t)|}\right\} \tag{4.12}$$

where

$$F(T) = \sup_{[0,T)} |\phi(t)|/(a_o - T\sup_{[0,T)} |\phi(t)|) \tag{4.13}$$

<u>Remark</u> Similar results hold for $\sup_{[0,T)} |\Psi(t)|$ and $\sup_{[0,T)} |\dot{\Psi}(t)|$, of course, under analogous assumptions on $\psi(t)$ and the series defining $\Psi(t)$, e.g., we require that $\sup_{[0,T)} |\psi(t)| < b_o/T$; the constant $F(T)$ appearing in (4.11), (4.12) would, in this case, be replaced by

$$G(T) = \sup_{[0,T)} |\psi(t)|/(b_o - T\sup_{[0,T)} |\psi(t)|) \tag{4.14}$$

In recalling the above lemma we have been motivated by a desire to replace the sufficient condition represented by (4.10) by a condition which involves only the basic memory functions $\phi(t)$, $\psi(t)$ specified in the constitutive relations (1.16a), (1.16b). To this end we note that the equations defining $\Phi(t)$ in terms of $\phi(t)$ and $\Psi(t)$ in terms of $\psi(t)$ imply, respectively, that

$$\Phi(t) + \frac{1}{a_o}\phi(t) = -\frac{1}{a_o}\int_0^t \phi(t-\tau)\Phi(\tau)d\tau \tag{4.15a}$$

$$\Psi(t) + \frac{1}{b_o}\psi(t) = -\frac{1}{b_o}\int_0^t \psi(t-\tau)\Psi(\tau)d\tau \tag{4.15b}$$

From (4.15a) and (4.15b) we immediately obtain

$$\Phi(0) = -\frac{1}{a_o}\phi(0),\ \Psi(0) = -\frac{1}{b_o}\psi(0) \tag{4.16}$$

and thus (4.10) can only be satisfied if $\phi(0) > 0$. Directly from (4.15b) we now compute that

$$\dot{\Psi}(t) + \frac{1}{b_o}\dot{\psi}(t) = -\frac{1}{b_o}\psi(0)\Psi(t) - \frac{1}{b_o}\int_0^t \psi_t(t-\tau)\Psi(\tau)d\tau \tag{4.17a}$$

$$\ddot{\Psi}(t) + \frac{1}{b_o}\ddot{\psi}(t) = -\frac{1}{b_o}\psi(0)\dot{\Psi}(t) - \frac{1}{b_o}\dot{\psi}(0)\Psi(t) - \frac{1}{b_o}\int_0^t \psi_{tt}(t-\tau)\Psi(\tau)d\tau \tag{4.17b}$$

Therefore,

$$\ddot{\Psi}(0) \equiv T(0) = -\frac{1}{b_o}(\ddot{\psi}(0) + \psi(0)\dot{\Psi}(0) + \dot{\psi}(0)\Psi(0)) \tag{4.18}$$

However, from (4.16) and (4.17a),

$$\dot{\Psi}(0) = -\frac{1}{b_o}\dot{\psi}(0) - \frac{1}{b_o}\psi(0)\Psi(0) = -\frac{1}{b_o}\dot{\psi}(0) + \frac{1}{b_o^2}\psi^2(0) \tag{4.19}$$

Combining (4.16_2) and (4.19_2) with (4.18) we have, finally,

$$T(0) = -\frac{1}{b_o}\left(\frac{1}{b_o^2}\psi^3(0) - \frac{2}{b_o}\psi(0)\dot{\psi}(0) + \ddot{\psi}(0)\right) \tag{4.20}$$

The left-hand side of (4.10) now assumes the form

$$\frac{1}{a_o^2}\phi(0) - \frac{\omega^2}{b_o}\left|\frac{1}{b_o^2}\psi^3(0) - \frac{2}{b_o}\psi(0)\dot{\psi}(0) + \ddot{\psi}(0)\right| \tag{4.21}$$

We now turn our attention to the right-hand side of (4.10). Directly from (4.17b) we obtain

$$\dot{T}(t) = -\frac{1}{b_o}(\psi^{(3)}(t) + \psi(0)T(t) + \dot{\psi}(0)\dot{\Psi}(t) + \ddot{\psi}(0)\Psi(t) + \int_0^t \psi_{ttt}(t-\tau)\Psi(\tau)d\tau) \tag{4.22}$$

Also,

$$\sup_{[0,T)}|T(t)| \le \frac{1}{b_o}\Big[\sup_{[0,T)}|\ddot{\psi}(t)| + |\psi(0)|\sup_{[0,T)}|\dot{\psi}(t)| + (|\dot{\psi}(0)| + T\sup_{[0,T)}|\ddot{\psi}(t)|)\sup_{[0,T)}|\Psi(t)|\Big] \tag{4.23}$$

while, by (4.22),

$$\sup_{[0,T)} |\dot{T}(t)| \le \frac{1}{b_o}[\sup_{[0,T)} |\psi^{(3)}(t)| + \psi(0) \sup_{[0,T)} |T(t)| \tag{4.24}$$

$$+ \dot{\psi}(0) \sup_{[0,T)} |\dot{\Psi}(t)| + (|\ddot{\psi}(0)| + T \sup_{[0,T)} |\psi^{(3)}(t)|) \sup_{[0,T)} |\Psi(t)|]$$

If we substitute for $\sup_{[0,T)} |T(t)|$ in (4.24) from (4.23) we obtain an estimate of the form

$$\sup_{[0,T)} |\dot{T}(t)| \le A \sup_{[0,T)} |\Psi(t)| + B \sup_{[0,T)} |\dot{\Psi}(t)| + C \tag{4.25}$$

where, the constants A, B, C are given by

$$A = \frac{1}{b_o} [T \sup_{[0,T)} |\psi^{(3)}(t)| + |\ddot{\psi}(0)| + \frac{|\psi(0)|}{b_o} (|\dot{\psi}(0)| + T \sup_{[0,T)} |\psi^{(2)}(t)|)]$$

$$B = \frac{1}{b_o}[|\dot{\psi}(0)| + \frac{\psi^2(0)}{b_o}]$$

$$C = \frac{1}{b_o}[\sup_{[0,T)} |\psi^{(3)}(t)| + \frac{1}{b_o}|\psi(0)| \sup_{[0,T)} |\psi^{(2)}(t)|]$$

As a result of the estimate (4.25), the right-hand side of the inequality (4.10) is bounded above by the expression

$$\omega^3 T(A \sup_{[0,T)} |\Psi(t)| + B \sup_{[0,T)} |\dot{\Psi}(t)| + C) + \frac{\omega T}{a_o} \sup_{[0,T)} |\dot{\Phi}(t)|, \tag{4.26}$$

which, in view of the preceding lemma, is itself bounded above by

$$D \equiv \omega^3 T[AG(T) + \frac{BG(T)}{T}(1 + T \frac{\sup_{[0,T)} |\dot{\psi}(t)|}{\sup_{[0,T)} |\psi(t)|}) + C] \tag{4.27}$$

$$+ \frac{\omega F(T)}{a_o} (1 + T \frac{\sup_{[0,T)} |\dot{\phi}(t)|}{\sup_{[0,T)} |\phi(t)|})$$

provided $\sup\limits_{[0,T)} |\phi(t)| < \frac{a_o}{T}$ and $\sup\limits_{[0,T)} |\psi(t)| < \frac{b_o}{T}$.

From (4.27), the definitions of the constants A, B, C, (4.13), and (4.14), it is clear that

$$\mathcal{D} = \mathcal{D}(\omega, T, a_o, b_o, |\psi^{(i)}(0)|, \sup_{[0,T)} |\phi^{(j)}(t)|, \sup_{[0,T)} |\psi^{(k)}(t)|) \tag{4.28}$$

with $i = 0,1,2$, $j = 0,1$, and $k = 0,1,2,3$. Thus, $\mathcal{D}$ is computable once Ω, $T > 0$, and the constitutive relations (1.16a), (1.16b) are specified. Thus (4.1) will be satisfied provided

$$\frac{1}{a_o^2} \phi(0) - \frac{\omega^2}{b_o} \left| \frac{1}{b_o^2} \psi^3(0) - \frac{2}{b_o} \psi(0)\dot{\psi}(0) + \ddot{\psi}(0) \right| \geq \mathcal{D} \tag{4.29}$$

We offer below an example of the kind of considerations which are involved in verifying that (4.29) is satisfied.

<u>Example</u> In the constitutive equations (1.16a), (1.16b) we take

$$\phi(t) = e^{-Kt}, \ \psi(t) = e^{-t} \tag{4.30}$$

where $K > 0$ is arbitrary; for the sake of convenience we set $T = 1$. The region $\Omega \subseteq R^3$ (and hence the embedding constant ω) are left arbitrary at this point as are the constants a_o, b_o. From (4.30) we have

$$\phi(0) = \sup_{[0,1)} |\phi(t)| = 1, \ \sup_{[0,1)} |\dot{\phi}(t)| = K \tag{4.31}$$

and

$$\sup_{[0,1)} |\psi^{(k)}(t)| = 1, \ k = 0,1,2,3 \tag{4.32a}$$

$$\psi(0) = \ddot{\psi}(0) = 1, \ \dot{\psi}(0) = -1 \tag{4.32b}$$

Therefore, the constants A, B, C in (4.25) are given by

$$A = \frac{2}{b_o}\left(1 + \frac{1}{b_o}\right), \quad B = C = \frac{1}{b_o}\left(1 + \frac{1}{b_o}\right) \tag{4.33}$$

Also, if $a_o > 1$, $b_o > 1$, then from (4.13) and (4.14)

$$F(1) = \frac{1}{a_o - 1}, \quad G(1) = \frac{1}{b_o - 1} \tag{4.34}$$

Combining our results it follows that (4.29) will be satisfied if a_o, b_o, and ω are chosen so as to satisfy

$$\frac{1}{a_o^2} - \frac{\omega(1+K)}{a_o(a_o-1)} > \omega^3 \frac{(b_o+1)}{b_o^2} \cdot \frac{b_o+3}{b_o-1} + \frac{\omega^2}{b_o}\left(\frac{1}{b_o^2} + \frac{2}{b_o} + 1\right) \tag{4.35}$$

As b_o must be restricted to satisfy $b_o > 1$, the right-hand side of (4.35), which we denote as $\sigma(b_o,\omega)$, is clearly positive. Thus, in order for (4.35) to be satified for an arbitrary $a_o > 1$, ω must satisfy

$$\omega = \omega_K < \frac{1}{1+K}\left(1 - \frac{1}{a_o}\right) < \frac{1}{1+K} \tag{4.36}$$

If we now choose Ω so that (4.36) is satisfied and define

$$\tilde{\sigma}(a_o, \omega_K) = \frac{1}{a_o^2} - \frac{\omega_K(1+K)}{a_o(a_o-1)}$$

then (4.35) becomes

$$\tilde{\sigma}(a_o, \omega_K) > \sigma(b_o, \omega_K) \tag{4.37}$$

But

$$\lim_{b_o \to +\infty} \sigma(b_o, \omega) = 0 \text{ (for any } \omega > 0) \tag{4.38}$$

and thus it is clear that for an arbitrary $a_o > 1$ and $\omega = \omega_K$ defined by

(4.36), the inequality (4.35) will be satisfied if b_o is chosen sufficiently large. We summarize our results in the following lemma:

Lemma Consider the holohedral isotropic dielectric material which is defined by the constitutive relations

$$\underset{\sim}{D}(\underset{\sim}{x},t) = a_o \underset{\sim}{E}(\underset{\sim}{x},t) + \int_0^t e^{-K(t-\tau)} \underset{\sim}{E}(\underset{\sim}{x},\tau)d\tau \tag{4.39a}$$

$$\underset{\sim}{H}(\underset{\sim}{x},t) = b_o \underset{\sim}{B}(\underset{\sim}{x},t) + \int_0^t e^{-(t-\tau)} \underset{\sim}{B}(\underset{\sim}{x},\tau)d\tau \tag{4.39b}$$

where $K > 0$ and $a_o > 1$ are arbitrary and $(\underset{\sim}{x},t) \in \Omega \times [0,1)$ with $\Omega \subseteq R^3$ chosen so that the bedding constant ω, defined by the inclusion map of H_o^1 into L_2, satisfies (4.36). If $\underset{\sim}{D}(\underset{\sim}{x},t) = \underset{\sim}{0}, (x,t) \in \partial\Omega \times [0,1)$, then there exists a constant $\Gamma > 1$ such that the operator $\underset{\sim}{M}(t)$, defined by (2.14b), satisfies the basic hypotheses (4.1) whenever $b_o \geq \Gamma$.

5. Relation to Previous Estimates for Holohedral Isotropic Dielectrics

In the present paper we have considered a special case of nonconducting holohedral isotropic dielectric response under the assumption of zero past history, i.e., $\underset{\sim}{E}(\tau)=0$, $\underset{\sim}{B}(\tau)=0$, $-\infty < \tau < 0$; our constitutive relations were, therefore, of the form (1.16a), (1.16b); using a logarithmic convexity argument we then derived growth estimates for the time evolution of the components of the electric displacement field in a dielectric which conforms to these constitutive hypotheses. In a recent work [11] we have derived different estimates for a closely related problem. Namely, we consider in [11] a holohedral isotropic material dielectric of the type (1.15a), (1.15b) with $a_\nu=0$, $b_\nu=0$, $\nu>1$ but with past history of the form

$$
(5.1)\qquad
\begin{aligned}
\underset{\sim}{E}(\underset{\sim}{x},t) &= \begin{cases} \underset{\sim}{0} & , -\infty < t < -t_h \\ \underset{\sim}{E}_h(\underset{\sim}{x},t), & -t_h \le t < 0 \end{cases} \\
\underset{\sim}{B}(\underset{\sim}{x},t) &= \begin{cases} \underset{\sim}{0} & , -\infty < t < -t_h \\ \underset{\sim}{B}_h(\underset{\sim}{x},t), & -t_h \le t < 0 \end{cases}
\end{aligned}
$$

where $t_h > 0$ is a given positive constant and $\underset{\sim}{E}_h$, $\underset{\sim}{B}_h$ satisfy appropriate smoothness assumptions on $\Omega \times (-t_h,0)$. The constitutive hypotheses in [11] then take the form

$$
(5.2)\qquad
\begin{aligned}
\underset{\sim}{D}(\underset{\sim}{x},t) &= a_0\, \underset{\sim}{E}(\underset{\sim}{x},t) + \int_{-t_h}^{t} \phi(t-\tau)\underset{\sim}{E}(x,\tau)d\tau \\
\underset{\sim}{H}(\underset{\sim}{x},t) &= b_0\, \underset{\sim}{B}(\underset{\sim}{x},t) + \int_{-t_h}^{t} \psi(t-\tau)\underset{\sim}{B}(\underset{\sim}{x},\tau)d\tau
\end{aligned}
$$

on $\Omega \times (-t_h,T)$, and, in place of the evolution equations (2.1) considered in the present work, we obtain, under the additional assumption that $\underset{\sim}{D}_h(\underset{\sim}{x},-t_h) = \underset{\sim}{0}$, uniformly on Ω, the evolution equations

$$
\frac{\partial^2 D_i}{\partial t^2} + \Psi(0)\frac{\partial D_i}{\partial t} + \dot{\Psi}(0)[D_i - \hat{c}_0\, \delta_{ik}\delta_{jl}\frac{\partial^2 D_k}{\partial x_j \partial x_l}]
$$
$$
+ \int_{-t_h}^{t} (\ddot{\Psi}(t-\tau)D_i(\tau) - \frac{b_o}{a_o}\,\Phi(t-\tau)\delta_{ik}\delta_{jl}\frac{\partial^2 D_k(\tau)}{\partial x_j \partial x_l})\, d\tau = 0
$$

for $i = 1,2,3$ with $\hat{c}_o \equiv b_o/a_o\dot{\Psi}(0)$. The same Hilbert space formalism used in the present work then leads in [11] to consideration of abstract initial-history value problems of the form (2.17),(2.18) but with $\beta(t) \equiv 0$ and with the integral operator defined on $[-t_h,T)$ instead of $[0,T)$. The basic differences, however, between [11] and the present work are as follows: In [11] we consider initial-history value problems corresponding to varying initial displacement fields and varying past histories, i.e.,

$$(5.4)\quad \begin{cases} \underset{\sim}{u}^{\alpha}_{tt} + \Gamma\underset{\sim}{u}^{\alpha}_{t} - \underset{\sim}{N}\,\underset{\sim}{u}^{\alpha} + \int_{-t_h}^{t} \underset{\sim}{K}(t-\tau)u^{\alpha}(\tau)d\tau = \underset{\sim}{0}, \quad 0 \le t < T \\ \underset{\sim}{u}^{\alpha}(0) = \alpha\, \underset{\sim}{u}_o,\ \underset{\sim}{u}^{\alpha}_{t}(0) = \underset{\sim}{v}_o \qquad \{ {\alpha > 0 \atop \underset{\sim}{u}_o, \underset{\sim}{v}_o \in H_+ }) \\ u^{\alpha}(\tau) = U(\tau),\ -t_h \le \tau < 0 \end{cases}$$

and

$$(5.5)\quad \begin{cases} \underset{\sim}{u}^{\beta}_{tt} + \Gamma\underset{\sim}{u}^{\beta}_{t} - \underset{\sim}{N}\,\underset{\sim}{u}^{\beta} + \int_{-t_h}^{t} \underset{\sim}{K}(t-\tau)\underset{\sim}{u}^{\beta}(\tau)d\tau = \underset{\sim}{0}, \quad 0 \le t < T \\ \underset{\sim}{u}^{\beta}(0) = \underset{\sim}{u}_o,\ \underset{\sim}{u}^{\beta}_{t}(0) = \underset{\sim}{v}_o \quad (\beta > 0) \\ \underset{\sim}{u}^{\beta}(\tau) = g(\beta)\underset{\sim}{U}(\tau),\ -t_h < \tau < 0 \end{cases}$$

where g is monotonically increasing on $[0,\infty)$. The basic aim of the work in [11] is not to derive growth estimates for the time evolution of $||\underset{\sim}{u}(t)||$ but rather to derive lower bounds for $\sup\limits_{(-t_h,T)} ||\underset{\sim}{u}^{\alpha}||_+$ ($\sup\limits_{(-t_h,T)} ||\underset{\sim}{u}^{\beta}||_+$) in terms of $\alpha(\beta)$ and the data of the problem: the conditions (3.2a),(3.2b) in the present work are weakened, in [11] to simply

$$(5.6)\qquad -<\underset{\sim}{v}, \underset{\sim}{K}(0)\underset{\sim}{v}> \ \ge 0,\ \forall\, \underset{\sim}{v} \in H_+$$

and the a priori condition that $\underset{\sim}{u} \in N$ (a class of bounded peturbations of the kind prescribed in §1) is dropped in [11] as logarithmic convexity is not employed to derive the desired estimates. Additional assumptions are made, however, in [11] relative to the data and the integral operator; namely,

$$(5.7)\quad \begin{cases} \int_0^{\infty} ||\underset{\sim}{K}(\tau)||_{L_s(H_+,H_-)} d\tau < \infty,\ \int_0^{\infty} ||\underset{\sim}{K}_{\tau}(\tau)||_{L_s(H_+,H_-)} d\tau < \infty \\ \int_{-t_h}^{0} ||\underset{\sim}{U}(\tau)||_+ d\tau < \infty \\ <\underset{\sim}{u}_o, \underset{\sim}{v}_o> \ > 0,\ <\underset{\sim}{u}_o, \underset{\sim}{N}\underset{\sim}{u}_o> \ > 0, \text{ and} \\ <\underset{\sim}{u}_o \int_{-t_h}^{0} \underset{\sim}{K}(-\tau)\underset{\sim}{U}(\tau)d\tau > \ < 0. \end{cases}$$

For the initial-history value problem (5.4) we then have the following result in [11]: Let $\underset{\sim}{u}^{\alpha}$ be a strong solution to (5.4) with

$$(5.8)\qquad \begin{aligned} &||\underset{\sim}{u}_o||^2 \le \frac{2}{\Gamma}<\underset{\sim}{u}_o,\underset{\sim}{v}_o> \\ &T > \frac{1}{\Gamma}\ln\left(\frac{2<\underset{\sim}{u}_o,\underset{\sim}{v}_o>}{2<\underset{\sim}{u}_o,\underset{\sim}{v}_o>-\Gamma||\underset{\sim}{u}_o||^2}\right)\end{aligned}$$

Then for each $\alpha > ||\underset{\sim}{v}_o||/<\underset{\sim}{u}_o,\underset{\sim}{N}\underset{\sim}{u}_o>^{\frac{1}{2}}$

$$(5.9)\qquad \sup_{[-t_h,T)} ||u^{\alpha}||_+ \ge \left[\frac{|<\underset{\sim}{u}_o,\int_{-t_h}^{o}\underset{\sim}{K}(-\tau)\underset{\sim}{U}(\tau)d\tau>|}{\omega\,\Sigma_T}\right]^{\frac{1}{2}}\sqrt{\alpha}$$

where

$$(5.10)\qquad \begin{aligned}\Sigma_T = \tfrac{1}{2}||N||_{L_s(H_+,H_-)} &+ \int_o^{\infty}||K(\tau)||_{L_s(H_+,H_-)}d\tau \\ &+ T\int_o^{\infty}||\underset{\sim}{K}_{\tau}(\tau)||_{L_s(H_+,H_-)}d\tau\end{aligned}$$

A similar result follows for the problem (5.5), with varying past history, under analogous assumptions. The basic idea behind the proof of the estimate (5.9) is as follows: Assume that (5.9) is false for some parameter value $\overline{\alpha} > ||\underset{\sim}{v}_o||/<\underset{\sim}{u}_o,\underset{\sim}{N}\underset{\sim}{u}_o>^{\frac{1}{2}}$ and show that $F_{\overline{\alpha}}(t)=||\underset{\sim}{u}^{\overline{\alpha}}(t)||^2$ satisfies the differential inequality

$$(5.11)\qquad F_{\overline{\alpha}}F_{\overline{\alpha}}''-(\overline{\alpha}+1)F_{\overline{\alpha}}''\ge -\Gamma F_{\overline{\alpha}}F_{\overline{\alpha}}',\quad 0\le t<T$$

which, in turn, implies that

$$(5.12)\qquad F_{\overline{\alpha}}^{\overline{\alpha}}(t) \ge F_{\overline{\alpha}}^{\overline{\alpha}}(0)[1-(1-e^{-\Gamma t})\overline{\alpha}F_{\overline{\alpha}}'(0)/\Gamma F_{\overline{\alpha}}(0)]^{-1}$$

The bracketed expression in (5.12) vanishes at

$$(5.13)\qquad t_{\infty}\equiv\frac{1}{\Gamma}\ln\left(\frac{2<\underset{\sim}{u}_o,\underset{\sim}{v}_o>}{2<\underset{\sim}{u}_o,\underset{\sim}{v}_o>-\Gamma||\underset{\sim}{u}_o||^2}\right)$$

and $t_\infty < T$ by virtue of the hypothesis (5.8). Thus $\sup_{(-t_h,T)} ||u^{\bar{\alpha}}||_+ = +\infty$ and via the embedding of H_+ into H this implies that $\sup_{(-t_h,T)} ||u^{\bar{\alpha}}|| = +\infty$ contradicting the assumption that

$$\sup_{(-t_h,T)} ||u^{\bar{\alpha}}||_+ \left[\frac{|\langle u_o, \int_{-t_h}^{o} K(-\tau)U(\tau)d\tau\rangle|}{\omega\, \Sigma_T}\right]^{\frac{1}{2}} \sqrt{\bar{\alpha}}$$

and, thus, establishing (5.9). Estimates of the type (5.9) can be very useful in terms of deriving estimates for physical parameters which enter the definition of the integral operator; in this vein we refer to a recent work [12] on Maxwell-Hopkinson dielectrics where estimates of the type (5.9) have been shown to lead to bounds for constitutive parameters appearing in the memory functions of such materials.

In a more recent work [13] initial-history boundary value problems associated with (5.3) have been reconsidered with a view toward deriving asymptotic lower bounds on the norms of the electric displacement vector when the operators in the equivalent initial-history value problem do not satisfy the requisite coerciveness conditions that imply asymptotic stability [14]. In fact, it is shown, in [13], that solutions $\underset{\sim}{u} \in N^*$ of the present abstract initial-history value problem ($N^* = \{\underset{\sim}{v} \in C([-t_h,\infty); H_0^1) \mid \sup_{[-t_h,\infty)} ||\underset{\sim}{v}||_{H_0^1} \le N\}$ for some $N > 0$) satisfy the differential inequality

$$\text{(5.14)} \qquad FF'' - \left(\frac{\beta+1}{2\beta+1}\right) F'^2 \ge -\Gamma FF', \quad F = ||\underset{\sim}{u}(t)||^2_{L_2}$$

for any $\beta > 0$, $0 \le t < \infty$, provided $E(0) = \frac{1}{2}||\underset{\sim}{v}_o||^2_{L_2} - \langle u_o, \underset{\sim}{N}\underset{\sim}{u}_o\rangle_{L_2} < 0$ with $|E(0)| > \frac{3}{2}\,\omega\, N^2[\,||K||_{L_1[0,\infty)} + ||\hat{K}||_{L_1[0,\infty)}]$ where we assume that (5.6) holds, and in addition, that

$$(5.15)\begin{cases} K(t) \equiv \|\underset{\sim}{K}(t)\|_{L_s(H_0^1,H^{-1})} \text{ satisfies } K(\cdot)\in L_1[0,\infty) \\ \hat{K}(t) \equiv \int \|\underset{\sim}{K}_t\|_{L_s(H_0^1,H^{-1})} d\tau \text{ satisfies} \\ \quad \hat{K}(\cdot) \in L_1[0,\infty) \text{ with } \hat{K}(0) = 0 \end{cases}$$

The differential inequality (5.14) then yields the estimate

$$(5.16) \qquad \lim_{t\to+\infty} \|\underset{\sim}{u}(t)\|^2_{L_2} \geq \|\underset{\sim}{u}_o\|^2_{L_2} \exp\left(\frac{2 < \underset{\sim}{u}_o, \underset{\sim}{v}_o >_{\underset{\sim}{L}_2}}{\Gamma \|\underset{\sim}{u}_o\|^2_{\underset{\sim}{L}_2}}\right)$$

so that $\lim\limits_{\Gamma\to+\infty} \lim\limits_{t\to+\infty} \|\underset{\sim}{u}(t)\|^2_{L_2} \geq \|\underset{\sim}{u}_o\|^2_{L_2}$. In fact, the sharper estimate

$$(5.17) \qquad \|\underset{\sim}{u}\|^2_{L_2} \geq \|\underset{\sim}{u}_o\|^2_{L_2} \left[1 + \left(\frac{2(1-\lambda)<\underset{\sim}{u}_o, \underset{\sim}{v}_o>_{\underset{\sim}{L}_2}}{\Gamma\|\underset{\sim}{u}_o\|^2_{\underset{\sim}{L}_2}}\right)(1-e^{-\Gamma t})\right]^{\frac{1}{1-\lambda}}$$

is shown to obtain in [14] for all $t > 0$ and any λ, $\frac{1}{2} < \lambda < 1$. Thus the L_2 norm of $\underset{\sim}{u}$ is bounded from below as $t \to +\infty$ even as the damping becomes arbitrarily large.

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Let $T > 0$ be an arbitrary real number and H, H_+ real Hilbert spaces with $H_+ \subseteq H$ algebraically and topologically and H_+ dense in H. Let H_- be the dual of H_+ via the inner product of H and denote by $L_s(H_+,H_-)$ the space of symmetric bounded linear operators from H_+ into H_-. We prove that the evolution of the electric displacement field in a simple class of holohedral isotropic

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) dielectrics can be modeled by an abstract initial-value problem of the form

$$\underset{\sim}{u}_{tt} - \alpha\underset{\sim}{u}_t - \underset{\sim}{L}\underset{\sim}{u} + \int_0^t \underset{\sim}{M}(t-\tau)\underset{\sim}{u}(\tau)d\tau = \beta(t)\underset{\sim}{u}_o,\ 0 \le t < T$$

$$\underset{\sim}{u}(0) = \underset{\sim}{u}_o,\ \underset{\sim}{u}_t(0) = \underset{\sim}{u}_1 (\underset{\sim}{u}_o, \underset{\sim}{u}_1 \in H_+)$$

where $L \in L_S(H_+,H_-)$, $\underset{\sim}{M}(t) \in L^2([0,T); L_S(H_+,H_-))$, $\beta(t) \in C^1([0,T))$, and α is an arbitrary (non-zero) real number. By employing a logarithmic convexity argument we derive growth estimates for solution of the above system which lie in uniformly bounded classes of the form

$$N = \{\underset{\sim}{u} \in C^2([0,T); H_+) \mid \sup_{[0,T)} ||\underset{\sim}{u}||_{H_+} \le N\}$$

for some $N > 0$; our results are derived under a variety of assumptions concerning α, $\beta(t)$, and the initial data (without making any definiteness assumptions on the operators L or M(t), $0 \le t < T$) and are used to obtain growth estimates for the electric displacement field $\underset{\sim}{D}(\underset{\sim}{x},t)$ in rigid dielectrics which satisfy constitutive relations of the form

$$\underset{\sim}{D}(\underset{\sim}{x},t) = a_o\underset{\sim}{E}(\underset{\sim}{x},t) + \int_0^t \phi(t-\tau)\underset{\sim}{E}(\underset{\sim}{x},\tau)d\tau$$

$$\underset{\sim}{H}(\underset{\sim}{x},t) = b_o\underset{\sim}{B}(\underset{\sim}{x},t) + \int_0^t \psi(t-\tau)\underset{\sim}{B}(\underset{\sim}{x},\tau)d\tau$$

where $\underset{\sim}{E}$, $\underset{\sim}{H}$, $\underset{\sim}{B}$ are the usual electromagnetic field variables, $(\underset{\sim}{x},t) \in \Omega \times [0,T)$, $\Omega \subseteq R^3$ is a bounded region with smooth boundary $\partial\Omega$, a_o and b_o are positive constants; and ϕ, ψ are non-negative monotonically decreasing functions of t.